THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirtieth regular meeting of the Society was held in New York City on Saturday, October 27, 1906, the following thirty-three members being in attendance:

Professor G. A. Bliss, Professor E. W. Brown, Dr. W. H. Bussey, Professor F. N. Cole, Miss E. B. Cowley, Dr. W. S. Dennett, Professor T. S. Fiske, Miss Ida Griffiths, Mr. S. A. Joffe, Professor Edward Kasner, Professor E. O. Lovett, Professor T. E. McKinney, Professor James Maclay, Professor Max Mason, Professor Richard Morris, Professor W. F. Osgood, Professor James Pierpont, Mr. H. W. Reddick, Professor L. W. Reid, Dr. R. G. D. Richardson, Miss S. F. Richardson, Miss I. M. Schottenfels, Professor Charlotte A. Scott, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor J. H. Tanner, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams, Professor J. E. Wright, Professor J. W. Young.

President W. F. Osgood presided at the morning session, Ex-President T. S. Fiske at the afternoon session. The Council announced the election of the following persons to membership in the Society: Professor A. F. Carpenter, Hastings College; Dr. H. M. Dadourian, Yale University; Mr. T. E. Gravatt, Pennsylvania State College; Rev. A. S. Hawkesworth, Allegheny, Pa.; Mr. H. R. Higley, Pennsylvania State College; Dr. Mario Kiseljak, Fiume, Hungary; Dr. Emanuel Lasker, New York, N. Y.; Professor Ernest Lebon, Lycée Charlemagne, Paris; Dr. R. L. Moore, Princeton University; Mr. W. P. Russell, Pomona College; Professor J. H. Scarborough, State Normal School, Warrensburg, Mo.; Mr. L. P. Siceloff, Columbia University; Professor Cyparissos Stephanos, University of Athens. One application for membership in the Society was received.

A list of nominations of officers and other members of the Council was adopted and ordered placed on the official ballot for the annual election at the December meeting. A committee was appointed to audit the Treasurer's accounts for the current year.

Professor W. F. Osgood tendered his resignation from the Editorial Committee of the *Transactions*, finding it impossible to assume the burdens of the office. The vacancy was filled by the appointment of Professor H. S. White.

The following papers were read at this meeting:

(1) Miss S. F. RICHARDSON: "Note on poristic systems of polygons."

(2) Professor R. D. CARMICHAEL: "Multiply perfect numbers of four different primes."

(3) Dr. ARTHUR RANUM: "On Jordan's linear congruence groups."

(4) Professor BEPPO LEVI. "Geometrie projettive di congruenza e geometrie projettive finite."

(5) Professor CHARLOTTE A. SCOTT: "Note on regular polygons."

(6) Professors MAX MASON and G. A. BLISS: "Some problems of the calculus of variations in space with variable end points."

(7) Professor EDWARD KASNER: "Note on the transformations of dynamics."

(8) Professor G. A. MILLER: "Groups of order p^m containing exactly $p+1$ abelian subgroups of order p^{m-1}."

(9) Professor G. A. MILLER: "The groups in which every subgroup is either abelian or hamiltonian."

Professor Levi's paper was communicated to the Society through Professor Osgood. In the absence of the authors, the papers of Professor Carmichael, Dr. Ranum, Professor Levi, and Professor Miller were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Miss Richardson's paper appears in full in the present number of the BULLETIN.

2. In this paper Professor Carmichael shows that there are but two multiply perfect numbers of (only) four different primes, and that these are $2^5 \cdot 3^3 \cdot 5 \cdot 7$, of multiplicity 4; and $2^9 \cdot 3 \cdot 11 \cdot 31$, of multiplicity 3.

3. Dr. Ranum proves that if G is any Jordan group, mod p^a, p being prime, and if G_i $(i = 0, 1, \cdots, a)$ is the invariant subgroup whose matrices are congruent to identity, mod p^i, then, except for certain low values of p, every matrix in G of

period $p^i (i = 0, \cdots, a-1)$ is contained in M_{a-i}, but not in M_{a-i+1}, and conversely; moreover every matrix of period p^i $(i = 0, \cdots, a)$ is commutative with every matrix of period p^{a-i}. These properties of Jordan groups are derived for the sake of their application to the more general linear congruence groups given in a paper entitled "The group of classes of congruent matrices, etc.," presented at the summer meeting of September, 1906. The great majority of the latter groups are thus shown to be not only new linear groups, but abstractly new, *i. e.*, they are not simply isomorphic with any Jordan groups of the same degree, or subgroups thereof.

4. In a note published in the *Transactions* (April, 1906) Drs. Veblen and Bussey have characterized the spaces consisting of a finite number of points in which a projective geometry exists. Professor Levi recalls that he has already occupied himself with spaces which include these as a particular case, in a paper on the "Fondamenti della metrica projettiva" (*Memorie* of the Turin Academy, series 2, volume 54, 1904). The coördinates of the points of these spaces are classes of numbers and rational expressions with rational (or entire) coefficients of a certain system (finite or not) of symbols, coresidual according to a determined irreducible modulus M. In these spaces a projective geometry exists which the author calls one of congruence.

Professor Levi indicates further how for such geometries the theorem of von Staudt is changed. After showing that, in any projective geometry, a transformation ϕ which converts harmonic groups into harmonic groups and leaves fixed the points 0, 1/0, 1 is characterized by the equality

$$\phi[f(\xi, \eta, \zeta, \cdots)] = f[\phi(\xi), \phi(\eta), \phi(\zeta), \cdots],$$

where f is a rational function, he demonstrates that, if the number of the points of a right line is finite and the modulus M does not contain numbers, the study of the transformation ϕ corresponds to the study of the birational transformations with rational coefficients of the variety which represents the modulus M in itself. If, on the contrary, to the modulus M belongs a prime number p and the question is precisely about finite geometries in which the abscissas of the points of a right line belong to a Galois field $G(p^n)$, instead of the theorem of Staudt

the following can be enunciated: Determine any succession whatever $n', n'', n''', \cdots, n$ of divisors of n, with $n' > 1$, and let each of them be equal to the preceding multiplied by a prime. To this succession of numbers $n^{(k)}$ one can get to correspond a succession of Galois fields $G(p^{n^{(k)}})$ contained in $G(p^n)$ and each containing the preceding, and a series of transformations ϕ characterized by the fact that each of them leaves fixed the points whose abscissas belong to one of these fields $G(p^{n^{(k)}})$, but removes the points of the next. The points belonging to one of these fields remain fixed as soon as one of them, not belonging to the preceding, remains fixed. To make sure that a ϕ comes to the identity it is necessary to verify at the most that it has as many fixed points, each rationally independent of those formerly considered, as there are prime divisors of the number n.

5. Miss Scott gave constructions for the regular pentagon, heptagon and nonagon by means of rectangular hyperbolas cutting a given circle at four vertices of the desired polygons; and from these deduced constructions for four vertices of the polygon on a given rectangular hyperbola.

6. The problem of finding a curve $y = y(x)$, $z = z(x)$, joining two given fixed points and minimizing an integral

$$I = \int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) dx$$

has been discussed according to recent methods by several writers, but comparatively little seems to have been done on the problem in which the end points are allowed to vary on surfaces or curves. In the paper of Professors Mason and Bliss the necessary conditions for a minimum when the end points are variable, analogous to those in the plane, and also sufficient conditions, have been derived. The problem in space has peculiarities essentially different from those of the corresponding problem in the plane on account of the fact that the end points may vary not only on curves but also on surfaces. The theory of congruences of curves plays an important part because the family of extremals transversal to a curve or surface is itself a congruence. A feature of the paper is the use of the parameter representation.

7. Appell has shown that the system of trajectories of a particle moving in a (positional) field of force is converted by any projective transformation into the system of trajectories corresponding to some other field of force. Professor Kasner showed, synthetically and analytically, that no other contact transformations possess this property.

8. Professor Miller's first paper appears in full in the present number of the BULLETIN.

9. Professor Miller's second paper is devoted to an extension of the results in the paper on "Non-abelian groups in which every subgroup is abelian" published in volume 4 of the *Transactions*. The principal theorems may be stated as follows: There is one and only one group of order 2^m which involves operators whose orders exceed four and satisfies the additional conditions that every subgroup is either abelian or hamiltonian and that at least one subgroup is hamiltonian. If every subgroup of a group of order 2^m, $m > 4$, is either abelian or hamiltonian and if it contains at least one hamiltonian subgroup, the entire group is hamiltonian. If a group contains at least one hamiltonian subgroup and if all its subgroups are either abelian or hamiltonian, it is the direct product of the hamiltonian group of order 2^m and an abelian group of odd order, unless it is the group of order 24 which does not contain a subgroup of order 12. There are only two non-hamiltonian groups which contain at least one hamiltonian subgroup and whose other subgroups are either abelian or hamiltonian.

F. N. COLE,
Secretary.

THE STUTTGART MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held at Stuttgart September 17–20, 1906, forming a section of the seventy-eighth convention of the Deutsche Naturforscher und Aerzte. The following papers were presented:

(1) O. BLUMENTHAL, Aachen: "Integral transcendental functions and Picard's theorem" (report).

(2) A. PRINGSHEIM, Munich: "Fourier's integral theorem."

(3) G. FABER, Karlsruhe: "Series of Legendre's polynomials."

(4) O. PERRON, Munich: "Singular points on the circle of convergence."

(5) F. HARTOGS, Munich: "Recent investigations of analytic functions of several variables" (report).

(6) P. STÄCKEL, Hanover: "Power series in several variables."

(7) D. HILBERT, Göttingen: "The nature and purposes of the theory of integral equations."

(8) E. HILB, Erlangen: "An extension of Klein's oscillation theorem."

(9) M. KRAUSE, Dresden: "On functions of real variables."

(10) P. KOEBE, Göttingen: "On the conformal mapping of multiply connected plane regions."

(11) F. MEYER, Koenigsberg: "The application of the generalized euclidean algorithm to the formation of resultants."

(12) P. SCHAFHEITLIN, Berlin: "On the theory of Bessel's functions."

(13) A. SCHOENFLIESS, Königsberg: "Report on the theory of point manifoldnesses."

(14) G. HESSENBERG, Grunewald: "Involution for transfinite ordinal numbers."

(15) G. LANDSBERG, Breslau: "On total curvature."

(16) K. ROHN, Leipzig: "Linear construction for cubic curves."

(17) TH. SCHMID, Vienna: "Constructive treatment of the axis complex."

(18) H. WIENER, Darmstadt: "Models for twisted curves of the third order."

(19) C. JUEL, Copenhagen: "Non-analytic curves in three dimensions."

(20) R. MÜLLER, Brunswick: "Determination of the pole in motion of a plane system at branching points."

(21) C. RUNGE, Göttingen: "Graphical solutions of differential equations."

(22) R. MEHMKE, Stuttgart: "New mechanisms for solution of dynamical problems."

(23) A. WAGENMANN, Stuttgart: "Mathematical theory of evolution."

1. Professor Blumenthal's paper was devoted to functions of infinite order. Reference was made to a new generalization of

Picard's theorem by Rémoundos (Thèse, Paris, 1906). The speaker's results are to appear in a volume of the Collection de monographies, dirigée par Borel.

2. In comparison with the thorough investigations of the limits within which Fourier's series is valid, the treatment of Fourier's integral seems somewhat inadequate. In addition to integrability and conditions which are necessary for the series, the integral theorem requires also that the function satisfy certain complicated conditions at infinity. Professor Pringsheim undertakes to replace the latter by simple sufficient conditions by carrying over Dirichlet's condition (generalized by DuBois-Reymond and C. Jordan) to intervals of the form $(A, +\infty)$, $(-A, -\infty)$. For this purpose he designates as quasi-monotone a function which is single-valued and finite in an interval (A, B) and can be expressed as the sum of two finite, monotone parts. The condition is question may then be formulated: At infinity (*i. e.*, for $x > A$ and $x < -A$ where A is an arbitrary positive quantity) $f(x)$ must converge towards zero monotone or quasi-monotone. The proof is to appear in the *Annalen*, also a more detailed account in the *Jahresbericht*.

3. Dr. Faber proved the following proposition: Denoting Legendre's polynomials by $P_\nu(x)$, the singular points β of the function defined by the series $\sum_{\nu=0}^{\infty} a_\nu P_\nu(x)$ are connected with the singular points of the function $\sum_{\nu=0}^{\infty} a_\nu x^\nu$ by the relation $\beta = \frac{1}{2}(\alpha + \alpha^{-1})$.

4. Dr. Perron generalized the following two propositions:

I. If a function $\sum a_n x^n$ admits on the circle of convergence only one singular point ρ, and if this is a rational pole, then a_n/a_{n+1} approaches ρ as a limit when n becomes infinite (Vivanti-Gutzmer, Theorie der eindeutigen analytischen Funktionen, page 397).

II. If there exists a limit $\rho = \lim_{n=\infty} (a_n/a_{n+1})$, then ρ is a singular point, although not necessarily a rational pole nor the only singular point on the circle of convergence. (Hadamard: La série de Taylor, page 25.)

Theorems I and II admit the following generalizations:

I*a*. If the singular points on the circle of convergence are all rational poles $\rho_1, \rho_2, \cdots, \rho_k$ of order $m_1, m_2, \cdots, m_k$ respectively and the greatest m occurs p times, then there exist p numbers $\xi_1, \xi_2, \cdots, \xi_p$ such that

$$\lim \frac{a_n + \xi_1 a_{n+1} + \cdots + \xi_p a_{n+p}}{max\,|a_n, a_{n+1}, \cdots, a_{n+p}|} = 0$$

and the roots of the equation $\rho^p + \xi_1\rho^{p-1} + \cdots + \xi_p = 0$ are precisely the p poles with the highest degree of multiplicity.

I*b*. With the hypotheses of I*a*, if $\rho^{p-1} + \eta_1\rho^{p-2} + \cdots + \eta_{p-1}$ is a divisor of $\rho^p + \xi_1\rho^{p-1} + \cdots + \xi_p$, the quotient

$$\frac{|a_n + \eta_1 a_{n+1} + \cdots + \eta_{p-1} a_{n+p-1}|}{max\,|a_n, a_{n+1}, \cdots, a_{n+p-1}|}$$

after a fixed value of n remains greater than a certain positive number σ.

II*a*. If p is the least number for which there exists a set of numbers $\xi_1, \xi_2, \cdots, \xi_p$ as in I*a*, and if, $\rho^{p-1} + \eta_1\rho^{p-2} + \cdots + \eta_{p-1}$ being a divisor of $\rho^p + \xi_1\rho^{p-1} + \cdots + \xi_p$, the quotients

$$\frac{|a_n + \eta_1 a_{n+1} + \cdots + \eta_{p-1} a_{n+p-1}|}{max\,|a_n, a_{n+1}, \cdots, a_{n+p-1}|}$$

after a certain value of n remain greater than a positive σ, then (1) the roots of the equation $\rho^p + \xi_1\rho^{p-1} + \cdots + \xi_p = 0$ all have the same value; (2) this absolute value is equal to the radius of of convergence; (3) the roots are singular points.

5. In connection with Weierstrass's lemma and the proofs by Poincaré and Cousin that a single-valued function meromorphic in the finite plane can be expressed as the quotient of two integral functions, Dr. Hartogs referred to a paper by Hahn in the *Wiener Monatshefte* in which the investigation of the zero-manifoldnesses of analytic functions of several variables is completely worked out and the problem of resolving integral functions into prime factors is solved. The investigation of those manifoldnesses which consist of the singular points of an analytic function is advanced on the one hand by ascertaining the general form of the region of convergence (Fabry, *Comptes rendus*, 1902; Faber, *Mathematische Annalen*, 61) and on the other by developing the analytic functions of x and y in series of the form $\sum f_\nu(x) \cdot y^\nu$ (Hartogs, *Annalen*, 62). Blumenthal has studied (*Annalen*, 57) those manifoldnesses which are common to the zero-manifoldnesses of several functions.

6. Professor Stäckel showed that in many cases the Cauchy-Weierstrass inequalities can be advantageously replaced by inequalities similar in form but simpler in principle and for which it is not necessary to know the course of the function in the complex plane, but only the absolute values of the terms of the series. In this investigation of the power series for a single variable it is only necessary to assume that it converges for a value of the variable other than zero, but the corresponding assumption for power series of several variables is not sufficient. Here it is necessary to require unconditional convergence. The question whether the unconditional convergence can be deduced from less assumptions has been considered by F. Hartogs (Dissertation and Habilitationsschrift).

The speaker communicated a more general proposition, which he had proved conjointly with Hartogs, according to which it is sufficient to know that the simple series obtained by rearranging the multiple series can be made to converge uniformly in a region including the origin by a single rearrangement. Whether the conditions can be still further restricted remains undecided. A detailed account will appear in the *Jahresbericht*.

8. By aid of the theory of integral equations Dr. Hilb obtains extensions of Sturm's theorem and Klein's oscillation theorem for the equations

$$\frac{d^2f}{dx^2} + k\phi(x)\cdot f(x) = 0 \tag{1}$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k[\phi(x) - \phi_1(y)]\, f(x,y) = 0 \tag{2}$$

respectively, if in (1) within a segment of the x-axis $\phi(x)$ is sometimes positive, sometimes negative, and if in (2) $\phi(x)$ is now greater, now less than $\phi_1(y)$. Sturm's theorem then becomes: To every oscillation number m correspond a positive and a negative value for which a particular function exists that vanishes m times within the segment. Klein's oscillation theorem becomes: To every oscillation pair m, n correspond two values (positive and negative) for which a solution of (2) exists that can be expressed in the form $h(x)\cdot g(y)$, where $h(x)$ vanishes m times and $g(y)$ n times within the segments. By an extension of the method which Dr. Hilb has developed in an article "Über die Reihentwickelungen der Potentialtheorie," to appear

in the next number of the *Annalen*, it follows that all the particular functions of (2) which vanish on the boundary of the rectangle formed by the two segments are identical with products $h(x) \cdot g(y)$. Hence we obtain the development in terms of such products.

9. Following some investigations of Borel, Professor Krause showed how continuous functions can be developed in uniformly convergent series of rational integral functions. He connected this with the known expressions for continuous functions in convergent series of rational integral functions and showed that under certain conditions uniformly convergent series can be assigned to the given functions.

10. Dr. Koebe proved the proposition that every triply connected plane region can be mapped conformally on a single-leaved region bounded by three circles which do not cut nor touch one another. He also gave a method of solving the corresponding problem for all n-ply connected single-leaved regions which are symmetric with respect to the real axis and have the property that every boundary cuts the real axis. The paper will appear in the *Jahresbericht*.

11. Professor Meyer extends Euclid's algorithm to rational integral functions $r_0(\lambda), r_1(\lambda), \cdots, r_{\kappa-1}(\lambda)$. The simultaneous vanishing of the $\kappa - 1$ last remainders is the necessary and sufficient condition for the existence of a variable common factor of the κ given polynomials, and the last remainder which does not vanish gives their greatest common divisor. Every one of these remainders can be expressed linearly in the given r's,

$$R_i = A_0^{(i)} r_0 + A_1^{(i)} r_1 + \cdots + A_{\kappa-1}^{(i)} r_{\kappa-1} (i = 1, 2, \cdots \kappa - 1). \quad \text{(I)}$$

An integral function of the coefficients of the r's which vanishes whenever two or more r's have a common root is called a partial resultant. Every partial resultant may be written

$$R = A_0 r_0 + A_1 r_1 + \cdots + A_{\kappa-1} r_{\kappa-1}. \quad \text{(II)}$$

If to the κ identities I and II we add the relation

$$0 = B_0 r_0 + B_1 r_1 + \cdots + B_{\kappa-1} r_{\kappa-1}, \quad \text{(III)}$$

and eliminate the r's, it follows that every partial resultant of

the given polynomials, multiplied by a suitable A, can be expressed linearly in terms of the last remainders $R_1, R_2, \cdots, R_{\kappa-1}$.

$$AR = A_1R_1 + A_2R_2 + \cdots + A_{\kappa-1}R_{\kappa-1}. \qquad \text{(IV)}$$

12. In a paper presented to the Berlin Mathematical Society, Professor Schafheitlin proved that the first root of the Bessel's function of the second kind $y_n(x)$ is greater than n and that these roots obey the law obtained by Schläfli for Bessel's functions of the first kind $J_n(x)$, namely, that each root, regarded as a function of n, increases with n. In the present paper it is proved that Schläfli's proposition is also true for all the roots of dJ_n/dx and for those of dy_n/dx which are greater than the first root of y_n. For small values of n these are all the roots of dy_n/dx. It appears that the first root of d^2y_n/dx^2, for small values of n, follows that of y_n but for $n > 7.5$, two roots of d^2y_n/dx^2 precede those of y_n. Thus the curve for $y_n(x)$, after cutting the x-axis to the right of $x = n$, is convex towards it, for small values of n. But for larger values of n it has two turning points before cutting the x-axis.

13. Professor Schoenfliess's report will appear in full in the *Jahresbericht*.

14. Professor Hessenberg defines the covering x of a finite part S_1 of a class S with elements of a class R as a finite covering of S with R, and denotes by (R/S) the class of all these finite coverings. If S and R are simply ordered, (R/S) can be simply ordered and its ordinal type (ρ/σ) possesses the properties of a power of $1 + \rho$ (Hausdorff, *Leipziger Berichte*, volume 26). If further R and S are well ordered, then by the process referred to (R/S) is also well ordered and (ρ/σ) is Cantor's power $(1 + \rho)^\sigma$ which possess the defining properties

$$a^1 = a, \quad a^{\beta+1} = a^\beta a, \quad \lim_\beta a^\beta = a^{\lim\beta}.$$

The author obtains a new proof of this last proposition.

15. Professor Landsberg pointed out that the theorems of Gauss and O. Bonnet on total curvature of surfaces can be, to a certain extent, carried over to the general problem of calculus of variations. For this purpose he classifies the problems which lead to the integral

$$s = \int \phi(x, y, x', y')dt$$

according to the degree m to which x' and y' enter into the equation of condition

$$T = \phi_{x'} \cdot \delta x + \phi_{y'} \cdot \delta y = 0$$

for the transformed position. If $m = 1$ and if the relation for the new position is an involution,

$$\phi = \sqrt{Ex'^2 + 2Fx'y' + Gy'^2}.$$

Then, depending on the sign of the discriminant, we obtain Gauss's form for the element of length or one in which the minimal curves are real. In both cases the concepts of geodetic curvature of a curve at a point, curvature of the field at a point, total curvature of a region can be so stated that an extension of the above two-dimensional domain is unnecessary. It follows that the change in the length of an arc on the surface due to a uniform transverse displacement of its points is equal to the mean value of the surface curvature along this curve; if the curve is contracted to an element of arc, the change is equal to the surface curvature itself. The Gauss-Bonnet theorem on curvatura integra of a portion of a surface has also its exact analogon in the case where the minimal curves are real.

17. If from every point P of a surface of the second order the normals p to the polar plane π are drawn, a one-to-one relation is established between the axis complex p and both the space P and the space π. From these relations Professor Schmid obtains the focal properties, constructions for the normals and for the principal centers of curvature, by projecting P and p from the center O, and the other vertices of the principal polar tetraedron on the corresponding opposite faces. The substance of the paper may be found in the *Sitzungsberichte* of the Vienna Academy for 1906 (volume 115, part II *a*).

18. Professor Wiener exhibited two models for space curves of the third order, regarding such a curve 1) as the common section of two cones with a common generatrix; 2) as enveloped by two planes which touch two conics with a common tangent. In both cases we meet with the osculating tetraedron determined by two points of the curve with their tangents and osculating planes. In 1) the vertices of the cones are the two

points of the curve; in 2) the planes of the conics are the osculating planes. Each model shows the curve by aid of its tangents in strings and the osculating tetraedron by wires. They complete the set of which twelve have been exhibited and will appear with these in the Teubner series.

19. Non-analytic curves can be classified according to their order and the class and number of singular points investigated by purely geometric methods. Professor Juel treated in this way curves of the fourth order obtained as intersections of two (non-analytic) cones of the second order and a curve of the nth degree lying on a hyperboloid and cutting every generating line of one set $n - 1$ times. For both classes the same characteristic numbers were obtained as in the algebraic cases. The question was also considered whether such curves have any real existence as non-analytic.

20. In the motion of a similarly deformable plane system the usual construction for the pole — depending on the tangents to the paths at three arbitrary points of the system * — fails when these tangents intersect on the circle through the three points. Professor Müller considered this case in the light of his results in a paper "Ueber die Krümmungsmittelpunkte der Bahnkurven in ebenen ähnlichveränderlichen Systemen" (*Zeitschrift für Mathematik und Physik*, volume 36, page 129) and showed how to determine the pole and that the system admits two initial motions from the position in question.

21. Professor Runge adapted Picard's method of successive approximations to the graphic solution of differential equations of the first order. The curve is first replaced by an approximating broken line so drawn that its integral wherever it cuts the curve in the direction of the x-axis agrees with the curve integral. The curve is here replaced by arcs of parabolas with axes parallel to the axes of x or y. In this way it is possible to determine the positions of the vertical parts of the broken line so that the areas above and below the curve shall be equal. On the surface $z = f(x, y) = dy/dx$, every curve $f(x, y) = \text{const.}$ determines a direction. A rough approximation to the required curve being drawn, a piece of it AB is taken small enough so that the above directions for AB do not include too large

* Burmester, Kinematik I, p. 867.

angles. This approximation is then made closer by using the values of f at points where AB cuts the curve $f(x, y) = \text{const.}$ If the second approximation is not close enough, the process is repeated.

23. Herr Wagenmann correlates successive steps in the theory of evolution with series $-\infty, \cdots -2, -1, 0, 1, 2, \cdots, \infty$ along three coördinate axes developing successively the ideas of motion, mass, the nebular hypothesis and evolution of living organisms and of civilization. He finds that his method leads to a monistic philosophy — in fact to a pan-monism.

A. B. FRIZELL.

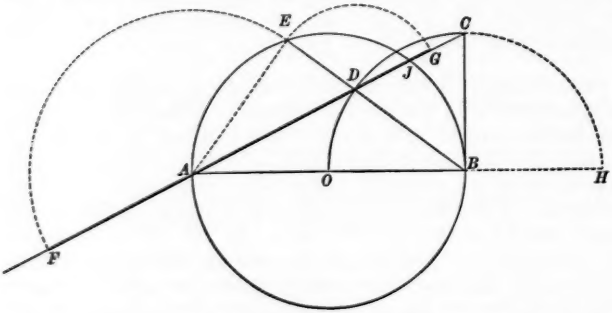
GÖTTINGEN,
November, 1906.

A NEW APPROXIMATE CONSTRUCTION FOR π.

BY MR. GEORGE PEIRCE.

GIVEN a circle with radius r and center at O; to find an approximate construction for πr.

Draw the diameter AOB and the tangent BC at right angles to it. Describe the arc ODC with radius r and center at B.



Draw the line AC cutting the arcs ODC and AB at D and J; also draw the line BDE through B and D cutting the given circle at E. Then $AD + 3DE = \pi r$ approximately.

Proof:

$$AC = \sqrt{(AB^2 + BC^2)} = r\sqrt{5}$$

$$AD = \frac{AO \cdot AH}{AC} = \frac{r \cdot 3r}{r\sqrt{5}} = \tfrac{3}{5}\sqrt{5}r,$$

$$JC = \frac{BC^2}{AC} = \frac{r^2}{r\sqrt{5}} = \tfrac{1}{5}\sqrt{5}r$$

$$DJ = AC - AD - JC = \tfrac{1}{5}\sqrt{5}r,$$

$$DE = \frac{AD \cdot DJ}{BD} = \frac{\tfrac{3}{5}\sqrt{5}r \cdot \tfrac{1}{5}\sqrt{5}r}{r} = \tfrac{3}{5}r,$$

$$AD + 3DE = \tfrac{3}{5}\sqrt{5}r + 3(\tfrac{3}{5}r) = 3.141641r.$$

By making use of the fact that in the triangle ABE

$$AE = \sqrt{(AB^2 - BE^2)} = \sqrt{(2r)^2 - (\tfrac{6}{5}r)^2} = \tfrac{8}{5}r = 2DE,$$

we can obtain a single line of the same length as $AD + 3DE$. We can therefore draw the arc EG with radius DE and center at D and the arc EF with radius AE and center at A. Then $AD + 3DE = AD + AE + DE = AD + FA + DJ = FG$.

There are many other approximate constructions for πr. A summary of those that have been worked out according to the method of geometrography is given below. A, B, C and D are to be found in the BULLETIN for January, 1902, page 137; E is in Cantor's Geschichte der Mathematik, volume 3, page 23; F is the construction given above.

	Author.	Δ	Without Square.				With Square.			
			S.	*E.*	Lines.	Circles.	*S.*	*E.*	Lines.	Circle
A	G. Peirce	+.0012	22	14	4	4	17	11	4	2
B	Kühn	+.0047	14	9	2	3	14	9	2	3
C	Lemoine	+.0030	21	13	2	6	20	13	2	5
D	Pleskot	−.00016	24	16	3	5	24	16	3	5
E	Kochansky	−.000060	33	20	6	7	23	13	6	4
F	G. Peirce	+.000048	24	15	4	5	19	12	4	3

Δ is the difference between the mechanically exact construction and π. *S* stands for simplicity and *E* for exactitude. For the technical meanings of these two words see the article in the BULLETIN for January, 1902. The lower these numbers are, the better the construction.

NOTE ON CONJUGATE POTENTIALS.

BY PROFESSOR O. D. KELLOGG.

(Read before the American Mathematical Society, September 3, 1906.)

IF $u(r, \vartheta)$ is a potential function of the unit circle and $v(r, \vartheta)$ its conjugate, and if $f(\vartheta)$ and $g(\vartheta)$ are the values approached by these functions as $r \doteq 1$, the following relations given by Hilbert * hold:

$$f(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi) \cot \tfrac{1}{2}(\vartheta - \phi)d\phi + \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi,$$

$$g(\vartheta) = -\frac{1}{2\pi}\int_0^{2\pi} f(\phi) \cot \tfrac{1}{2}(\vartheta - \phi)d\phi + \frac{1}{2\pi}\int_0^{2\pi} g(\phi)d\phi,$$

where by the integration symbols in the first terms of the right hand sides the Cauchy principal value is meant. They give, to within an additive constant, the boundary values of a potential in terms of its conjugate.

They have been established for the case that $f(\vartheta)$ and $g(\vartheta)$ are integrable throughout and are continuous at all but a finite number of points and possess derivatives subject to the same conditions.† Because of the interest attaching to them from their connection with potential theory, and as examples of integral equations of the first kind, ‡ it seems worth while to point out that they hold if only $f(\vartheta)$ and $g(\vartheta)$ are integrable from 0 to 2π and are such that

$$\int \frac{f(\vartheta_0 + \delta) - f(\vartheta_0)}{\delta} d\delta \quad \text{and} \quad \int \frac{g(\vartheta_0 + \delta) - g(\vartheta_0)}{\delta} d\delta$$

are convergent when extended over an interval including $\delta = 0$ in its interior and this for all but a finite number of values of

* Vorlesungen über Potentialtheorie, Göttingen, winter semester, 1901–02.

† See *Math. Annalen*, vol. 58, p. 442; also my dissertation for the doctorate: "Zur Theorie der Integralgleichungen und des Dirichlet'schen Prinzips," Göttingen, 1902, p. 17.

‡ Hilbert: "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Gött. Nachrichten*, 1904, p. 49. Encykl. d. Math. Wiss. II, A, 11, pp. 803, 816.

ϑ_0 in the interval from 0 to 2π.* To justify this assertion it will only be necessary to point out how a minor change in the article in the *Mathematische Annalen* or in the dissertation just referred to permits the generalization in question.

A slightly different notation is there used: instead of the arguments ϑ and ϕ, $s = \vartheta/2\pi$ and $t = \phi/2\pi$ are employed. The reasoning may remain unchanged except for the matter of showing that the integral

$$I = \int_{-\epsilon}^{+\epsilon} g(t) \frac{\partial}{\partial t} \log \rho \, (^{t}_{x,\,y}) dt$$

is a continuous function of x and y in the neighborhood of $x = 1$, $y = 0$ (*Mathematische Annalen*, page 445; dissertation, page 21). It will be sufficient to show that this integral vanishes with ϵ in a sufficiently small neighborhood of (1, 0).

To do this, let us use instead of x and y the coördinates r and s, where $x = r \cos 2\pi s$, $y = r \sin 2\pi s$. Then $\rho^2(^{t}_{r,\,s}) = 1 + r^2 - 2r \cos 2\pi(s - t)$, and we may write the integral $I = I_1 + I_2$, where

$$I_1 = g(s) \int_{-\epsilon}^{+\epsilon} \frac{\partial}{\partial t} \log \rho(^{t}_{r,\,s}) dt,$$

and

$$I_2 = \int_{-\epsilon}^{+\epsilon} [g(t) - g(s)] \frac{\partial}{\partial t} \log \rho(^{t}_{r,\,s}) dt.$$

The first of these gives

$$I_1 = g(s) \log \frac{\rho(^{\epsilon}_{r,\,s})}{\rho(^{-\epsilon}_{r,\,s})},\dagger$$

and, as we are at a point where $g(s)$ is continuous, this factor is finite, and if we restrict the point (r, s) to that neighborhood of (1, 0) given by $0 < 1 - r < \epsilon^2$, $|s| < \epsilon^2$, the logarithm will be found to vanish at least as fast as a constant times ϵ. Hence I_1 vanishes with ϵ.

I_2 may be written

$$\int_{-\epsilon}^{+\epsilon} \left[\frac{g(t) - g(s)}{t - s}\right] \frac{2\pi r \sin 2\pi(s - t)}{1 + r^2 - 2r \cos 2\pi(s - t)} \cdot (t - s) dt.$$

* These conditions are even less restrictive than those announced in my paper as read before the Society.

† This on the understanding that if $r = 1$ the Cauchy principal value is to be taken.

Now

$$\frac{2\pi r \sin 2\pi(s-t)}{1+r^2-2r\cos 2\pi(s-t)} \cdot (t-s)$$

is a continuous function of t for any fixed values of r and s in the above defined neighborhood of (1,0). This is at once evident when $r<1$, and becomes evident for $r=1$ when we give it the form

$$-\frac{\pi(s-t)}{\sin \pi(s-t)}\cos \pi(s-t).$$

Hence we may infer that I_2 vanishes with ϵ, for

$$\frac{g(s)-g(t)}{s-t},$$

being continuous for $t \neq s$, and having a convergent integral even over an interval including $t=0$ by hypothesis, the convergence of I_2 may be made apparent by integrating by parts, and, being convergent, it vanishes with its limits $+\epsilon$ and $-\epsilon$.

To recapitulate, the integral

$$\int_0^1 g(t)\frac{\partial}{\partial t}\log \rho(_{r,s}^{t})dt,$$

which we wished to show continuous in r and s in the neighborhood of (1, 0), breaks up into two parts

$$\int_\epsilon^{1-\epsilon} \quad \text{and} \quad \int_{1-\epsilon}^{+\epsilon},$$

or J and I as we may denote them. As I vanishes with ϵ, as we have just shown, we may so restrict ϵ and r and s that $I<\delta/4$. Then the variation of I in the neighborhood of (1, 0) thus determined will be less than $\delta/2$. But ϵ being fixed, J is continuous, and hence r and s may be so restricted that its variation is less than $\delta/2$. Thus the variation of the whole integral is less than δ, and it is therefore continuous at the point (1, 0), and that for any manner of approach whatever within or upon the periphery of the circle.

The formulas are thus established under the given conditions.

UNIVERSITY OF MISSOURI,
October 4, 1906.

GROUPS OF ORDER p^m CONTAINING EXACTLY $p+1$ ABELIAN SUBGROUPS OF ORDER p^{m-1}.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, October 27, 1906.)

IF an abelian group of order p^m, p being any prime, contains exactly $p+1$ subgroups of order p^{m-1} it has just two invariants and vice versa. Since these groups are so well known, we shall confine ourselves, in what follows, to non-abelian groups of order p^m. It is known that in every non-abelian group of this order the number of abelian subgroups of order p^{m-1} is 0, 1, or $p+1$. The present paper confines itself to the last of these three possible cases. Since any non-cyclic group of order p^m is generated by two of its subgroups of order p^{m-1}, and two such subgroups have p^{m-2} common operators, it follows that *the necessary and sufficient condition that a non-abelian group of order p^m contain $p+1$ abelian subgroups of order p^{m-1} is that it contain p^{m-2} invariant operators.* These invariant operators form a characteristic subgroup and the corresponding quotient group is of type (1, 1).

While we shall consider only non-abelian groups in the present paper, yet the abelian subgroups of order p^{m-1} will enter largely into our discussions. It seems therefore desirable to state here a few fundamental theorems with respect to abelian groups in the form in which they will be used. *If all the invariants of an abelian group of order p^m are equal, all its operators of the same order are conjugate under its holomorph and vice versa.* Any abelian group is the direct product of abelian subgroups in which all the invariants are equal. The number of complete sets of conjugate subgroups of order p^{m-1} under the holomorph of any abelian group of order p^m is equal to the number of its different sets of equal invariants, since this number is equal to the number of the different sets of conjugate subgroups of order p under the holomorph. * *The independent generators of any subgroup of order p^{m-1} in an abelian group of order p^m may be so chosen that they are either contained in a possible set of independent generators of the entire group, or that*

* *Amer. Jour. of Math.*, vol. 22 (1900), p. 21.

all of them except one are contained in such a set while this one is the pth power of the other independent generator of this set.

The last theorem may be proved as follows: Since the number of distinct sets of conjugate subgroups of order p^{m-1} under the holomorph is equal to the number of the different sets of equal invariants of the group G it is only necessary to observe that the independent generators of one out of each such set of these conjugate subgroups can be chosen in the given manner. This may be done by constructing, in succession, the groups which result by using the pth power of one of the independent generators of G together with all its other independent generators, using the pth power of only one of the independent generators of the same order in G. The independent generators of the subgroups of lower order than p^{m-1} cannot always be chosen in this way nor is it always possible to select the independent generators of G in such a way that they become, when one of them is raised to the pth power, identical with the independent generators of a subgroup of order p^{m-1} which have been arbitrarily chosen. These statements will be illustrated in the following paragraph.

Let s_1, s_2 represent two independent commutative operators of orders p^α, p respectively and suppose $\alpha > 1$. It is impossible to choose the independent generator of the subgroup $\{s_1^p s_2\}$ of index p^2 under $\{s_1, s_2\}$ in such a manner that it either is contained in a possible set of independent generators of $\{s_1, s_2\}$ or that it is a power of such a generator. This illustrates the former of the two statements in question. If we take as the independent generators of a subgroup of index p under $\{s_1, s_2\}$, the two operators $s_1^p s_2$, s_2 the latter of these statements is illustrated, since it is impossible to select the independent generators of $\{s_1, s_2\}$ in such a manner as either to include $s_1^p s_2$, s_2 or to make $s_1^p s_2$ a power of such a generator. In fact neither of the operators $s_1^p s_2$, s_2 is a power of a larger operator contained in $\{s_1, s_2\}$.

Since a characteristic subgroup of an abelian group cannot involve any of its operators of highest order, it follows that the largest characteristic subgroup of any abelian group is composed of all its operators which are not of highest order. In particular, *the necessary and sufficient condition that an abelian group of order p^m contains a characteristic subgroup of order p^{m-1} is that it has only one invariant of highest order.*

§1. *General Properties.*

Let H represent one of the $p+1$ abelian subgroups of order p^{m-1} and let K represent the subgroup of order p^{m-2} which is composed of the invariant operators under the entire group G. The general method which will be used in what follows is to assume H as known and then to consider all the possible groups of order p^m which contain this H. The three most important steps are: 1) the choice of K; 2) the possible transformations of H by the operators of G which are not contained in H; 3) the total number of groups which transform H in a given manner. The details of construction are given in the article * entitled "A method of constructing all groups of order p^m."

From the theorem mentioned above it follows that K can be chosen in as many distinct ways as there are different sets of equal invariants in H. That is, the number of sets of subgroups of order p^{m-2} in H which are such that each set includes all those which are conjugate under the holomorph of H is equal to the number of sets of equal invariants in H. This number will be denoted by k. As the invariants of the k distinct subgroups which are successively represented by K may be obtained by raising one of those of H, in succession, to the pth power, while the others remain unchanged, no two of these k subgroups are simply isomorphic. In particular, when all the invariants of H are equal, K can be chosen in only one way; so that $k=1$ in this case.

We shall now consider the transformations of H by possible operators of G. In other words, we shall consider the number of different sets of operators of order p in I (the group of isomorphisms of H) which are such that each operator is commutative with all the operators of a particular K and that each set includes all the operators of I which are conjugate under the subgroup of I which transforms K into itself. As all these transformations of H may be obtained by making it isomorphic with one of its subgroups of order p and multiplying corresponding operators,† the problem is reduced to finding the number of sets of conjugate subgroups of order p in H. This number is either k or $k+1$, as the invariant of H which is p times the corresponding invariant of K is unique or is equal to

* *Amer. Jour. of Math.*, vol. 24 (1902), p. 395.
† BULLETIN, vol. 7 (1901), p. 350.

at least one other invariant of H. Under the holomorph of G the subgroups of order p in K may be contained in $k-1$ sets of conjugates, as will be seen later.

It remains to consider the third of the three steps mentioned above, viz., the construction of the total number of groups which transform H in a fixed manner. There is clearly one and only one such G which involves an operator t of order p that is not contained in H. As K can be selected in k different ways, the total number of G's which involve H and also operators of order p which are not contained in H is $k(k+1)-l$, where l is the sum of the numbers of the sets of equal invariants of H which contain only one invariant $+1$ or 0, according as there is at least one invariant which is equal to p or no such invariant.

Each of the $k(k+1)-l$ groups considered in the preceding paragraph contains at least p abelian subgroups which are similar to H whenever $p>2$. The invariants of the remaining abelian subgroup of order p^{m-1} are obtained by dividing one of those of H by p and adding to this quotient and the other invariants of H one which is equal to p. In the groups in which these invariants are equal to those of H the $p+1$ abelian subgroups of order p^{m-1} in G are similar. There are k such groups when H contains more than one invariant which is equal to p, $k-1$ when H contains only one such invariant, and there is no such group when the smallest invariant of H exceeds p. When $p=2$ it is still true that at least two of the abelian subgroups of order 2^{m-1} are similar but they are not necessarily similar to H.

According to the general theory of constructing the groups of order p^m, all of the other groups having the required property may be constructed by making H simply isomorphic with itself written in p distinct sets of letters and replacing t by itself written in the same systems, multiplied by a substitution which simply interchanges these systems and by an operator s from one of these systems. The number of different groups is equal to the number of distinct ways of choosing s. This choice can generally be made in $k+1$ ways, but in special cases there may be a smaller number of choices, as follows directly from the general theory. The main result which we desire to emphasize here is that since s is in K but is not a power of an operator of higher order contained in K, the remaining p abelian subgroups of order p^{m-1} are similar to each other except when just $p-1$ of them are similar to H, while

the other involves more operators whose order is p times that of s than H does. We are thus led to the main theorem of this section, which may be stated as follows: *If a non-abelian group of order p^m involves $p+1$ abelian subgroups of order p^{m-1}, at least p of these subgroups are similar to each other, and the entire group is conformal with an abelian group whenever $p>2$.*

Since a non-abelian group of order p^3 necessarily contains exactly $p+1$ subgroups of order p^2, the above theorem includes the theorem that a non-abelian group of order p^3 contains at least p similar subgroups of order p^2. It may also be stated that the preceding paragraph proves that the conformal non-abelian groups of order p^m which contain $p+1$ abelian subgroups of order p^{m-1} are not necessarily identical, while it is well known that abelian groups are identical whenever they are conformal.

§ 2. *Groups in Which the $p+1$ Abelian Subgroups of Order p^{m-1} are Similar to Each Other.*

Although these groups are included in the preceding section, yet it seems desirable to treat them somewhat more completely in view of the fact that they are of special interest. When all the invariants of H are equal to each other they are all equal to $p>2$, or to 4. In the former case there is one and only one group of order p^m, $m>2$. As all its operators are of order p, it is conformal with the abelian group of order p^m and of type $(1, 1, 1, \cdots)$. When all the invariants of H are equal to 4 there is one and only one group of order 2^{2n+1}, $n>0$. Since the commutator of order 2 in such a group is the square of a non-invariant operator of order 4, it is the direct product of the quaternion group and the abelian group of order $2^{n'}$ and of the type $(1, 1, 1, \cdots)$. These results are expressed in the following theorem: *If a non-abelian group of order p^m involves $p+1$ abelian subgroups of order p^{m-1} and if all the invariants of these subgroups are equal to the same number, then either the group is the direct product of the quaternion group and an abelian group of order $2^{n'}$ and of the type $(2, 2, 2, \cdots)$, or it is the direct product of the non-abelian group of order p^3 $(p>2)$ which involves no operator of order p^2 and the abelian group of order $p^{n'}$ and of type $(1, 1, 1, \cdots)$*

If the conditions of the preceding paragraph are made somewhat more general by merely requiring that the invariants of each one of the $p+1$ abelian subgroups be equal to each other, there are only two additional groups which satisfy these condi-

tions, viz., the octic group and the non-abelian group of order p^3 which involves operators of order p^2. These properties might be employed to give new definitions of these groups. For instance, the former of these two groups, which is so fundamental in elementary mathematics, may be defined as follows: the octic group is the only non-abelian group of order 2^m which contains three abelian subgroups of order 2^{m-1} such that the invariants of each subgroup are equal to each other, but not all these invariants are equal to the same number.

We shall now consider the case where the $p+1$ abelian subgroups of order p^{m-1} are similar to each other and where each of them involves more than one set of equal invariants. From the preceding section it follows that the invariants of H must satisfy at least one of the following two conditions: The smallest is either p or 4, or the ratio of some two is p. When the former of these conditions is satisfied, but not the latter, the invariants of K may be obtained from those of H by dropping one of the smallest ones, if these are equal to p, or by replacing a 4 by a 2. In the former case the number of possible G's, when $p>2$, is either k or $k-1$, according as H contains more than one or only one invariant which is equal to p. When $p=2$ there are always $k-1$ such G's. Finally, when the smallest invariants of H are equal to 4 there is just one such group involving a given H. This exhausts the possible cases where no two invariants of H have p for their ratio.

When H involves invariants whose ratio is p and also operators of order p or 4, the groups which have just been determined will still exist. The remaining possible groups are independent of whether H involves operators of order p or 4. When $p>2$ there are $k+\gamma$ groups for every two sets of equal invariants which are such that the ratio of the invariants of one set with respect to those of the other is p. The value of γ is -1, 0, or 1, according as each of these two sets includes only one invariant, one includes one while the other includes more than one, or each of them includes more than one.

When $p=2$ the remarks of the preceding paragraph remain true except when H involves invariants which are equal to 2 and also invariants which are equal to 4. In this special case it is only necessary to consider the possible groups where K does not involve all the operators of order 2, or where it involves one less invariant which is equal to 4 than H does. In the former case there are $k-1$ distinct G's. In the latter case there are

$k+2$, $k+1$, or k such G's. The number is $k+2$ when there is more than one invariant of each of values 2 and 4 in H. When there is only one invariant of one of these values and more than one of the other, the number of G's is $k+1$. Finally, there are only k such G's when H contains only one invariant of each of the values 2 and 4.

It may be added that in the study of all the possible non-abelian groups of order p^m which contain an abelian subgroup of order p^{m-1} it is especially desirable to know all of those groups which contain more than one such subgroup, as the other possible groups are distinct whenever they transform the abelian subgroup in different ways. When there is more than one abelian subgroup of order p^{m-1} in G, two such subgroups may be transformed differently by the remaining operators.

THE UNIVERSITY OF ILLINOIS,
September, 1906.

NOTE ON SYSTEMS OF IN- AND CIRCUMSCRIBED POLYGONS.

BY MISS S. F. RICHARDSON.

(Read before the American Mathematical Society, October 27, 1906.)

IN a paper read before the London Mathematical Society on March 12, 1874 (*Proceedings of the London Mathematical Society*, volume 5) Wolstenholme assumes two similar and similarly situated polygons of n sides, $ABC\cdots KLM$ and $abc\cdots klm$, and considers the conditions for an infinity of polygons which shall be inscribed in one of the similar polygons and circumscribed about the other.

He assumes that if ab meet AM in U and if am meet AB in V, then

$$AU/AM = AV/AB = k.$$

His solution finds $n-1$ values for k, that is, that there are $n-1$ points on AM, say, which may be taken as its intersection with ab, this point fully determining the polygon $abc\cdots klm$.

In particular he finds as the two solutions for the case $n=3$ that ab must divide AC in the ratio $\frac{1}{3}$ or in the ratio 1. In the first case the triangle abc becomes a point, the common

point of intersection of the lines drawn through the points of trisection of each side of ABC parallel respectively to the other two sides of the triangle. In the second case abc is a triangle circumscribing ABC, the vertices of ABC being the midpoints of the sides of abc.

For the case $n = 4$ his three solutions are that the side ab must divide the side AD in one of the ratios $\frac{1}{4}\cos^2 \frac{1}{8}\pi$, $\frac{1}{4}\cos^2 \frac{2}{8}\pi$, $\frac{1}{4}\cos^2 \frac{3}{8}\pi$, of which the middle one he says applies only to a parallelogram and reduces the quadrangle $abcd$ to a point while the other two give proper solutions and for squares give the theorem: "If $ABCD$ and $abcd$ be two concentric squares with their sides parallel and the side of one equal to the sum of the side and diagonal of the other, we can find an infinite number of quadrangles $A'B'C'D'$ such that A', B', C', D' lie on the sides AB, BC, CD, DA, and their sides $A'B'$, $B'C'$, $C'D'$, $D'A'$ pass through b, c, d, a respectively; or such that A', B', C', D' lie on the sides ab, bc, cd, da and their sides $A'B'$, $B'C'$, $C'D'$, $D'A'$ pass through D, A, B, C respectively."

He adds that in this case and for the parallelogram in general, the in- and circumscribed polygon can never, as far as he has investigated, be a convex polygon.

It will be of interest to compare the properties of the triangle and quadrangle shown by this solution with those properties shown by the synthetic solution of the general problem as proposed by Poncelet of determining the "lieu du sommet libre d'un polygone variable, dont les autres sommets parcourent des droites données, tandis que ses côtés pivotent sur des points fixes," and the "cas pour lesquels le lieu des sommets libres s'abaisse au premier degré."

Taking arbitrarily one of the polygons $ABC \cdots KLM$ and $n - 2$ consecutive vertices $b, c, \cdots, k, l$ of the other polygon, the problem of finding the remaining two vertices of the second polygon is the elementary problem of finding two centers a and m from which two projective point rows AB and LM may be projected into the same point row on a given line AM. The solution is known. The locus of the vertex a is the line MM' joining M to the point on AB which corresponds to M regarded as a point of the point row LM, and the locus of m is the line AA' joining A to the point on LM which corresponds to A regarded as a point of the point row AB, a pair of the required vertices being found as the intersections respectively of MM'

and AA' with the join of any third pair of corresponding points on AB and LM.

In the case of the triangle ABC, the point rows on AB and BC projected from the given vertex b of the triangle abc being perspective, the vertices a and c must either coincide with b or lie on a line through B. That is, the triangle abc reduces to a point or is circumscribed to ABC.*

Hence the *necessary and sufficient conditions that there be an indefinite number of triangles inscribed in one of two triangles and circumscribed about the other is that the two triangles be one inscribed in the other, the poles being the vertices of the circumscribed triangle.*

Wolstenholme's solution for the triangles is a special case of this solution.

In the case of the quadrilateral $ABCD$ and the given vertices b and c of the quadrilateral $abcd$, if the point rows on AB and CD projected respectively from the vertices b and c into the point row BC are perspective to each other, then the joins of A and B to their corresponding points on CD and the joins of C and D to their corresponding points on AB must all four meet in some point P, and the given vertices b and c must lie on some line through the point (AB, CD). They are respectively the intersections of this line with CP and BP. Similarly the vertices a and d must also lie on a line through the point (AB, CD). They may be the intersections of DP and AP with any second line through the point (AB, CD).

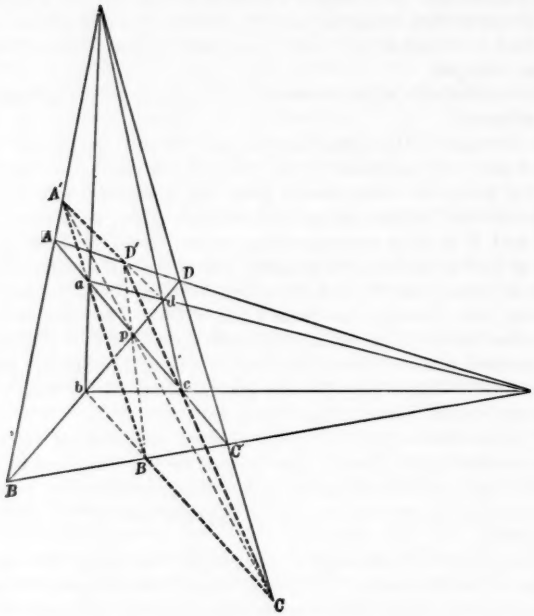
It follows that a sufficient condition for a system of in- and circumscribed quadrilaterals with respect to two quadrilaterals is that the two quadrilaterals be in homology, and that a pair of opposite sides of one and a pair of opposite sides of the other be concurrent.

The center of homology P, say, might be made the intersection of the diagonals of $ABCD$. And then the points d and c (or a and b) might chance to be collinear with the point (CB, DA) [assuming that by construction a and b and therefore b

* Mr. Cayley, by his solution of the following problem proposed by Mr. W. K. Clifford: Through the angles A, B, C, of a plane triangle straight lines Aa, Bb, Cc, are drawn. A straight line AR meets Cc in R; RB meets Aa in P; Pc cuts Bb in M; MA meets Cc in r; and so on. Prove that after going twice round the triangle in this way, we always come back to the same point. (Cayley's Collected Math. Papers, volume 5, p. 589) has shown that if the vertices of the inscribed triangle be made the poles there will be closure for all points if the projection be made twice round the triangle.

and c are collinear with the point (AB, DC)]. Then the point rows AB and DC would also be perspective and the points a and b would therefore also be collinear with the point (CD, DA). This case is represented in the diagram below.

Hence, *if $ABCD$ be any quadrilateral and if through the point (BA, CD) a line be drawn meeting the diagonals AC, BD in the points a, b, respectively, and if these points be joined to the point (BC, DA) by lines meeting the diagonals BD, AC in the*



points d, c, respectively, then the line dc passes through the point (BA, DC) and there exists an indefinite number of quadrilaterals inscribed in $ABCD$ and circumscribed about $abcd$ if AB be projected on BC, BC on CD, CD on DA, DA on AB from the vertices c, d, a, b, respectively, or from the vertices a, b, c, d, respectively. It is immaterial which of the two quadrilaterals be named $ABCD$.

Hence, also, *if $ABCD$ and $abcd$ be two concentric parallelograms similar and similarly placed, there exist with respect to them an indefinite number of in- and circumscribed quadrilaterals if AB be projected on BC, BC on CD, CD on DA, DA on AB from the vertices c, d, a, b, respectively, or a, b, c, d, respectively, or if ab be projected on bc, bc on cd, cd on da, da on ab from the vertices C, D, A, B, respectively, or from A, B, C, D, respectively.*

Moreover in this case *the in- and circumscribed quadrilaterals are all parallelograms;* for the point rows AB and CD as well as the point rows AD and BC are perspective, the center of projection for both pairs of point rows being the point P, the intersection of the diagonals of $ABCD$. Hence the diagonals of each of the variable quadrilaterals pass through P, since each diagonal joins a pair of corresponding points of perspective point rows whose center of projection is P. Since these diagonals are all bisected at P the variable quadrilaterals are parallelograms.

It follows from this proof that beside the system of in- and circumscribed quadrilaterals obtained by Wolstenholme with respect to the concentric squares referred to above, there is another system obtained by projecting AB on BC, BC on CD, CD on DA, DA on AB, from the vertices c, d, a, b, respectively, or from the vertices a, b, c, d, respectively, or by projecting ab on bc, bc on cd, cd on da, da on ab, from the vertices C,D,A,B, respectively, or from the vertices A, B, C, D, respectively, and that in this case the variable quadrilaterals are parallelograms instead of reentrant quadrilaterals.

If $ABCD$ be a quadrilateral and if through the point (AB, CD) lines be drawn parallel to BC and AD respectively and through the point (BC, AD) lines parallel to BA and CD respectively, these lines form a quadrilateral in homology with $ABCD$. Hence there is an infinite system of in- and circumscribed quadrilaterals with respect to these two quadrilaterals. The vertices of either may be used as poles.

If $ABCDEF\ldots$ be any polygon of an even number of sides and if a, b, c, etc., be respectively arbitrary points on the diagonals AC, BD, CE, etc., there will be closure for the points A and B if AB be projected on BC, BC on CD, CD on DE, etc., from the poles a, b, c, etc. respectively. Hence if any third point on AB be projected into itself arbitrarily, the intersections of the projecting lines with AC, BD, CE, etc.,

will be the vertices of a polygon with respect to which and $ABCDEF\ldots$ there is an indefinite number of in- and circumscribed polygons.

If $ABCDEF\ldots$ be any *regular* polygon of an even number of sides and if AC intersect FB and BD in a, b respectively, CE intersect BD and DF in c, d respectively, etc., then by the properties of the regular polygon there will be closure for the midpoint of AB (and so for all points of AB) if AB be projected on BC, BC on CD, CD on DE, etc., from the poles a, b, c, etc., respectively. Also there will be closure for the midpoint of AB and so for all points of AB if AB be projected on BC, BC on CD, etc., from the midpoints of AC, BD, CE, etc., respectively. If $ABC\cdots$ have an *odd* number of sides and the projections be made from poles corresponding to those indicated in these two theorems, there will be closure for all points if the projection be made twice round the polygon.

If $ABCDE$ be any pentagon and if the points (AC, BD), (BD, CE), (DA, EC), (BE, DA), (AC, EB) be named e, a, b, c, d respectively, then if $ABCDE$ and $abcde$ be regarded as two simple pentagons there will be a poristic system of pentagons if ab be projected on bc, bc on cd, cd on de, de on ea, ea on ab from the vertices B, C, D, E, A respectively. This may be proved by testing for closure when the vertices of $ABCDE$ or of $abcde$ are the points projected.

[Professor Morley has pointed out to me that the simple pentagon $ACEBD$ and any one of the variable pentagons constitute the ten-point, ten-line configuration of Desargues's two perspective triangles.]

VASSAR COLLEGE,
October, 1906.

HERMITE'S WORKS.

Oeuvres de Charles Hermite, publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. I. Paris, Gauthier-Villars, 1905. 8vo. xl + 498 pp.

I.

ON January 14, 1901, Charles Hermite passed away. A contemporary and zealous disciple of Gauss, Jacobi, and Dirichlet, the friend and generous rival of Cayley, Sylvester, and Brioschi,

his death snapped one of the few remaining links connecting the present with the heroic days of the last century.

Hermite was born December 24, 1822, at Dieuze, Lorraine. His parents being in comfortable circumstances, he was early sent to continue his studies at Paris. Here at the lycée Louis le Grand, he had the good fortune to come under the charge of Professor Richard who, as may be remembered, had taught Galois in the same class of mathématiques spéciales some twelve years before.* Hermite, like Galois, early manifested an extraordinary talent for mathematics. Neglecting the regular courses of study, he read with greatest ardor the masterpieces of Euler, Lagrange, Gauss, and Jacobi; and such was the precocity of his genius that while still at Louis le Grand he published two papers and had others well under way.

In 1842, Hermite entered the Ecole Polytechnique as sixty-eighth in his class. His love for the higher mathematics had left him little leisure to prepare for examinations; hence his poor standing. His stay at this famous school was, however, destined to be short. From birth Hermite had suffered from an infirmity of the right leg and had to use a cane. On this account it was now decided by the authorities that he should not be eligible to any of the government positions which are given to the graduates of the Ecole. Hermite, therefore, left at the end of the first year. Although brief, his attendance at the Ecole Polytechnique was not without effect on his further career. For, while here, he was encouraged by Liouville to compose the first of those remarkable letters to Jacobi (January, 1843) in which the genius of Hermite shone with such extraordinary luster.

A rapid and brilliant academic career would have been the natural reward for such exceptional talents; but honors and high position were apparently to be obtained then in only one way, a path full of bitterness and humiliation to the spirit of Hermite, who had all examinations *en horreur*. It was indeed necessary for him to descend from his lofty mathematical speculations which were opening up new fields of research and occupy himself at the age of twenty-four with the petty and irksome details of preparation for the examination leading to the degrees of bachelor of letters and of science, the *licence* and the *agrégation*.

* J. Pierpont, "Early history of Galois's theory of equations." BULLETIN, 2d Series, vol. 4, p. 335.

The first three he passed successfully. He still had the last and most trying of all before him when he fortunately received the position of examinateur d'admission and répétiteur d'analyse at the Ecole Polytechnique in 1848. These five years 1843–1848 were the Sturm und Drang period of Hermite's otherwise placid life. His appointment at the Ecole Polytechnique removed him from his trying and painful position apart from the university world, and placed him in regular standing among his friends and contemporaries. Although his advance was still by no means rapid, others with less talents receiving preference, the doubts and uncertainties which had beset his future and disturbed his meditations now melted away, and Hermite entered on his long and peaceful career with its course clearly marked out.

We close this brief sketch with a few important dates. From 1848 to 1850 he substituted at the Collége de France in place of Libri. In 1856 he was elected to the Institut. In 1862 a place as maître de conférence was created for him at the Ecole Normale. In the following year he was appointed to the responsible position of examinateur de sortie at the Ecole Polytechnique. In 1869 he became professor at this school and in 1870 at the Sorbonne. He was thus forty-seven years of age when he at length reached a position which his extraordinary talents had long claimed as their due. The professorship at the former institution he held till 1876; at the latter he remained in active service till 1897. Hermite was therefore 75 years of age when he closed his academic career.

As a lecturer at the Sorbonne, Hermite achieved a world-wide reputation. Nearly all the present generation of French mathematicians received instruction at his hands and the affectionate veneration of his former pupils was touchingly manifested on the occasion of his jubilee in 1892. Here is the tribute of one of his most gifted pupils, E. Borel: "C'est à la Sorbonne que j'ai suivi les leçons d'Hermite; c'est là que j'ai entendu cette parole si vivante exposer avec respect à la fois et avec amour les belles vérités de l'analyse. C'était un grand prêtre de la divinité du nombre qui nous en dévoilait les mystères redoutables et sacrés. Les questions les plus arides, les calculs en apparence les plus ingrats se transfiguraient, tant il avait l'intuition de leurs secrètes beautés. Quelques-uns peut-être ont eu, autant qu'Hermite, le pouvoir de faire comprendre

et admirer les mathématiques; nul n'a su les faire aimer aussi profondément que lui."*

II.

Let us now turn our attention to the volume under review. This in one respect will certainly prove the most interesting one in Hermite's works. In fact, it is in these early memoirs that we may observe his wonderful genius unfold and mature; it is here that we see the germs of those fruitful notions and principles evolve with which Hermite enriched the theory of numbers, the theory of invariants and of the elliptic and abelian functions. Our astonishment and admiration grow apace as we study the contents of the 37 memoirs which make up this volume and bring us down to the year 1858. At the beginning of his career of research, Hermite seems to have actually suffered from a wealth of ideas; they overwhelm him to such an extent that he cannot take the time to develop them in order and with leisure. Rapid sketches of the main results and principles, hasty aperçus of broad horizons and fruitful fields to be exploited later, are characteristics of many of these first papers.

At the period when Hermite as a young man of twenty began to strike out for himself, the arithmetic theory of the higher forms was just beginning to be studied. Gauss had already taken the first step in the *ternary* quadratic forms when treating the composition of binary quadratic forms in the Disquisitiones Arithmeticæ. Dirichlet and Eisenstein were ably carrying on the work, when Hermite appeared upon the scene with several new and powerful principles. Chief of these are: (1) The introduction of forms with *variable* coefficients; (2) the determination and systematic employment of the upper limits of the minimum values that certain forms can take on when the variables are restricted to integral values. By their aid Hermite attacked with marked success two fundamental problems in the theory of forms, viz: the proof that the number of classes of forms of a given kind is finite, and the determination of all the linear substitutions which leave a given form unchanged. The particular forms that Hermite here studied

* For further details we refer to the following:
G. Darboux, "La vie et l'oeuvre de Charles Hermite," *La Revue du Mois*, Paris, 1906; vol. 1, p. 37.
E. Picard, preface to the volume under review.
M. Noether, "Charles Hermite." *Math. Annalen*, vol. 55, p. 337.

are the quadratic forms in n variables, bilinear conjugate imaginary forms, and forms which are the product of n linear factors. While studying these profound questions, many are the applications he makes en route to problems of the most diverse nature. Such are, for example, the following: decomposition of primes into factors formed of the roots of unity; the generalization of continued fractions, leading to a criterion for cubic irrationalities, and also affording a simultaneous approximation to n given numbers; the demonstration that an analytic function of n variables cannot have more than $2n$ periods; the determination of the maximum number of fundamental units in cyclotomic bodies, the finiteness of the number of independent algebraic irrationalities belonging to the same discriminant, and the decomposition of a number as the sum of four squares. Perhaps the most notable of all is his application of the theory of quadratic forms to the theorems of Sturm and Cauchy relative to the number of roots of an algebraic equation in a given region.

It was in the theory of numbers that Hermite first won lasting fame, and the papers on this subject not only form the larger part of the present volume, but they also have a very real importance in the problems of the present. Have we not seen what wonderful developments they were capable of in the hands of Minkowski, and are not the bilinear forms in conjugate imaginaries playing today a prominent role in automorphic functions and linear groups?

However, it was not as a disciple of Gauss and Dirichlet that Hermite made his debut in the mathematical world, but as a zealous student of the writings of Jacobi. Sixteen years before (1827) Jacobi, then an obscure privat docent at Königsberg, penned those lines, now historic, to the venerable Legendre: "Un jeune géomètre ose vous presenter quelques découvertes faites dans la théorie des fonctions elliptiques, auxquelles il a été conduit par l'étude assidue de vos beaux écrits. . . ."

No doubt with these circumstances in mind, Liouville advised Hermite to communicate some of his results to Jacobi, who now was at the zenith of his fame at the University of Königsberg. The reception and generous recognition which this letter received at the hands of Jacobi encouraged Hermite to give a further account of his researches in a second letter. Jacobi thought so highly of them that they were published in his Opuscula Mathematica as well as in *Crelle's Journal* (volume 32,

1846). The first letter was indeed a tour de force for a young man scarce twenty years of age. In 1832 Jacobi had at last succeeded in discovering the inversion of the ultraelliptic integrals and two years later in his paper "De functionibus duarum variabilium," etc., he had given a few meager indications relative to their division and transformation. This was all that was then known. The whole subject was involved in obscurity and but few mathematicians of that day had any knowledge of it whatever. Jacobi might well be surprised therefore on receiving Hermite's letter containing a complete solution of the problem of division. His second letter treats the question of the transformation of these functions, but with less success. The letter is, however, noteworthy as containing the germs of the modern theory of theta functions of one variable, of order n and rational characteristics, and their application to the transformation theory of the elliptic functions.

Jacobi's reply to this letter terminates with the following noble and generous passage: "Ne soyez pas fâché, monsieur, si quelques-unes de vos découvertes se sont rencontrées avec mes anciennes recherches. Comme vous dûtes commencer par où je finis, il y a nécessairement une petite sphère de contact. Dans la suite si vous m'honorez de vos communications, je n'aurai qu'à apprendre."

The problem of transformation of the ultraelliptic functions left unfinished, as just remarked, required for its solution the introduction of new elements, the theta functions in two variables and the arithmetic properties of the abelian group defined by the linear transformation of the four periods. The former was furnished by the papers of Goepel and Rosenhain (1847–1851), the latter by Hermite's own long researches in the theory of numbers. Thus ten years later, in 1855, Hermite was able to give the complete solution of the problem in a memoir which he regarded in later years with just pride. Another important application of the theory of numbers is Hermite's determination of the twenty-fourth root of unity which enters in the linear transformation of the thetas, and whose value until then was unknown. This forms the last paper in the present volume; it is moreover the forerunner of his epoch-making memoirs on the elliptic modular functions, and their application to the solution of the quintic.

To complete our picture of the rich contents of this first volume we have yet to mention Hermite's early contributions

to the theory of invariants. This theory which Cayley and Sylvester had just begun to develop attracted him at first by its numerous and important relations with the higher arithmetic. Indeed, in both theories the linear transformations are of prime importance and the first invariants discovered are the determinants of quadratic forms. The present volume contains three great memoirs on binary forms, from the years 1854, 1856; their most interesting and original feature is the application Hermite makes of the principles he had already elaborated in the arithmetic theory of forms. In the profusion of results given in these memoirs we note the law of reciprocity, the existence of quadratic covariants for forms of all degrees except the biquadratic, the introduction of canonical forms, the complete system of forms for the quintic, the discovery of skew invariants, criteria for the reality of the roots of the quintic in terms of its invariants, the sextic resolvent of the quintic and the reduction of elliptic differentials of the first species to the canonical form now known as Weierstrass's.

In these investigations we see Hermite attach a particular importance to the forms of fifth degree. In fact at this time he was carrying on several lines of investigations which were to culminate in his memorable solution of the quintic. Two we have already mentioned; a third we have deferred until the present moment. Like so many of the foremost of his predecessors Hermite had been very early attracted to study the solution of the general equation of fifth degree. While still a pupil at Louis le Grand and ignorant of the work of Abel and Galois, he had published a demonstration of the impossibility of its solution by radicals, which now forms the second paper in the present volume. That Hermite still had this problem in mind we know from a letter of Borchardt, 1847. Thus when Puiseux published his memoir, 1850, on the permutation of the roots of an algebraic equation around the branch points, Hermite immediately made use of these results to develop the notion of the monodromic group of the equation and saw its application to the equations of division and transformation of the elliptic functions, 1851.

A few miscellaneous remarks in closing. As noticed above, the volumes are to appear in royal octavo and not in the customary quarto form. This first volume is furnished with a portrait of Hermite, at about the age of twenty-five; and has a preface by Picard on Hermite's scientific work. The editor

gracefully acknowledges the assistance rendered by the late Professor Stouff in checking and correcting by laborious computations the results of some of the memoirs. Many of the papers bear no date and none state the pages of the volume in which they originally appeared. A moment's reflection will convince the editor how important either of these data may be. Hermite sometimes refers in a general way to some of his earlier results and it would greatly help the reader if precise references were given to the pages of the present volume. Let us illustrate. In a paper on the equation for secular inequalities Hermite (page 481), refers to certain forms he introduced "comme on pourra le voir dans un de mes Mémoires publiés dans le *Journal de Crelle*, t. 47." The reader unfamiliar with Hermite's work will naturally turn to the table of contents to find the place of this memoir in the present volume. Here he will make the unpleasant discovery that the titles of the memoirs are unaccompanied by the name and volume of the journals in which they appeared. The reader will therefore have to turn to one title page after another, until he comes to page 193 where he meets a memoir from this volume of *Crelle*. Here he finds nothing about these forms. He looks further and discovers that there is another memoir also in *Crelle* 47. This is a long paper and it takes some time to ascertain that the forms in question are not there. He may now learn that there is still a third memoir in this same volume of *Crelle*, and here on page 237 he may at last find what he so long searched for.

Again, when Hermite refers to the results of others, it would often help the reader if exact reference were given to their collected works. For example, Hermite refers, page 484, to a paper of Jacobi's in *Crelle*, volume 34. This is a misprint, as the paper is to be found in volume 36, page 97, or in Jacobi's Werke, volume 2, page 173. Or again, on page 380, Hermite refers to results "que j'ai indiquées dans le Journal de M. Thomson." As relatively few persons will know what journal is meant, why not add a footnote that it is the *Cambridge and Dublin Mathematical Journal*, and give the precise reference to its place in the present volume, viz., page 301? On pages 79 *seq.* we meet a sign which certainly will be unintelligible to some readers. It is Cauchy's symbol for the residue of an analytic function, and is, we believe, entirely obsolete now.

We have noted the following misprints: Page 165, 8 lines from top, for 46 read 40; page 315, 10 lines from top, for

foudent read *fondent;* page 332, last line, letter missing; page 380, for *Thompson* read *Thomson*; page 449, a hyphen is missing in next to the last line, also capitalize d; page 473, for *Veierstrass* read *Weierstrass;* page 481, heading of page is wrong; page 498, 8 lines from bottom; for *snr* read *sur.*

JAMES PIERPONT.

PROJECTIVE DIFFERENTIAL GEOMETRY.

Projective Differential Geometry of Curves and Ruled Surfaces. By E. J. WILCZYNSKI. Leipzig, B. G. Teubner, 1906. viii + 295 pages.

THE present volume is the amplification and systematic development of the ideas originally presented in various papers by the author, published in the *American Journal, Transactions of the* AMERICAN MATHEMATICAL SOCIETY and *Mathematische Annalen.*

The work begins with a very brief resumé of the ideas of continuous groups, followed by a synopsis of the transformations of linear homogeneous differential equations, wherein Stäckel's theorem regarding the form of the transformations which leave such an equation invariant is generalized to apply to a simultaneous system of such equations. A fairly full discussion is given to the invariants and covariants of a single linear equation. After showing that every transformation which leaves the equation invariant is of the form

$$y = \lambda(x)\eta, \quad x = f(\xi),$$

the first transformation alone is treated at length, the functions of p_i, $p_i^{(k)}$ which remain invariant being designated as seminvariants; those of p_i, $p_i^{(k)}$, $y^{(l)}$ being called semi-covariants. By the second transformation, a function Ω of x is said to be invariant of weight m if

$$\Omega(x) = \frac{1}{(f')^m}\,\Omega(\xi).$$

An early application of these ideas is the derivation of the canonical form of the equation, in which the terms containing $y^{(n-1)}$ and $y^{(n-2)}$ are absent. The Lagrange adjoints are dis-

cussed and geometric applications made. Every solution of the adjoint furnishes an integrating factor of the equation.

Chapter III contains the first real application of the dominant idea of the book. Given a linear differential equation of order 3

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0, \tag{1}$$

the semi-covariants z, ρ are defined by

$$z = y' + p_1y, \quad \rho = y'' + 2p_1y' + p_2y.$$

The one linear invariant is

$$\theta_3 = P_3 - \tfrac{3}{2}P_2',$$

the P_i being the coefficients in the semi-canonical form (*i. e.*, lacking y''). The quadri-derivative of θ_3 (Forsyth) is

$$\theta_{3,1} = 6\theta_3\theta_3'' - 7(\theta_3')^2.$$

Now let three independent solutions of (1) be regarded as homogeneous point coördinates in the plane. As x varies, they will define an analytic curve. The only identically self-dual curves (every tangent associated with its own point of contact) are shown to be conics. The invariants θ_3 and $\theta_{3,1}$, given as arbitrary functions of x, define a plane curve projectively. When the point $(y_1, y_2, y_3) \equiv P_y$ describes a curve, P_z, P_ρ describe other curves, semi-covariantly related to C_y. The equations of the osculating conic (having contact with C_y of the fourth order at a given point) and osculating cubic are now derived. A pencil of C_3's can be found having eight point contact. The residual basis point is called the Halphen point. In particular, it may coincide with the other eight. Such a point is called a coincidence. The cubic now has a double point and no cubic of the pencil has proper contact of the eighth order.

An interesting discussion of C_z, C_ρ follows, including a number of anharmonic properties. If all the points of C_y are coincidence points, the curve is projectively equivalent to a logarithmic spiral whose parametric angle is 30°.

If $p_1 = p_2 = 0$, C_3 superosculates C_y. If this happens for all values of x, C_y is itself a C_3. The condition that this happens is expressed in terms of invariants, and a classification of elliptic, nodal, cuspidal cubics is expressed by them. If $\theta_3 = 1$, $\theta_{3,1} = 0$, C_y is C_3.

An application of elliptic functions is then made, showing that the coincidence points on a C_3 are grouped in triangles which are inscribed in and circumscribed about the curve. Finally, the condition that C_y be an anharmonic (or W-) curve is also derived, and various theorems established.

Many of these results were previously obtained by Halphen and others, but the present treatment is much more systematic and complete, and may be read without a knowledge of the previous memoirs on the subject.

These ideas of transformations and invariants of a single differential equation are extended in Chapter IV to a system of linear equations. The first general paper on this subject was published by the author in the *Transactions*, volume 2, pages 1–10. This chapter embodies that paper and pages 99, 104, 106–110, 118–125 are new. Particular attention is paid to two equations, each of the second order, on account of the later application of this system. The equations are reduced to their semi-canonical form and to their canonical form, then the complete system of invariants is derived. The functionally complete system consists of four forms and those resulting from these by repetition of the jacobian process. The system of covariants is also derived. Later a precise geometric meaning is given to each of these forms, and the specializations which appear when one or more of them identically vanishes.

The part of the book that is the essentially novel work of the author commences with Chapter V, foundations of the theory of ruled surfaces. The first paper appeared in the *Transactions*, volume 2, pages 343–362.

From the simultaneous equations of the second order

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0$$

we can define two twisted curves C_y, C_z, by letting x vary. The line (yz) will describe a ruled surface S which is projectively defined by the equations. At the torsal generators of S, the tangents to C_y, C_z intersect. By constructing the tangent planes to S at P_y, P_z we obtain the solutions of another system of linear equations, the adjoint of the given one. The two systems have the same invariants and seminvariants; they are identical if S is a quadric. When the equations are reduced to the semi-canonical form ($p_{12} = p_{21} = 0$), C_y, C_z are asymptotic

lines on S. The fundamental theorem is that when the four basic invariants are given as arbitrary functions of x, not identically zero, then S is uniquely determined, projectively considered. When three basic invariants of one system are equal to those of another, but the fourth negatively equal, the two surfaces are dualistic. If θ_4 is zero, S is identically self-dual, *i. e.*, every point is transformed into the tangent plane at that point and leaves the surface invariant.

The locus of the point through which a transversal can be drawn cutting four consecutive generators of S is the flecnode curve. It is not enveloped by the flecnodal tangents unless it be a straight line directrix. These tangents form another ruled surface, called the flecnode surface of S. As many of these properties are most easily expressed by means of line coordinates, a short resumé of line geometry of the first degree is introduced, then the ideas are applied to interpret the invariants and covariants derived before. Perhaps the most important is that the necessary and sufficient condition that a ruled surface be identically self-dual is that it belong to a non-special linear complex.

If the two branches of the flecnode curve coincide, or if the surface belongs to a special linear complex, it is not determined by means of the four fundamental invariants.

An important idea is that of the flecnode congruence, made up of the generators (of one system) of the osculating hyperboloids H of S. Its focal surface is shown to be the flecnode surface of S and various properties are established. If we put

$$\rho = 2y' + p_{11}y + p_{12}z, \quad \sigma = 2z' + p_{21}y + p_{22}z,$$

then when the line (yz) describes S, the line $(\sigma\rho)$ will describe another surface S', called the derivative of S as to x. It is contained in the flecnode congruence of S, one of its generators lying on the H of each generator of S. Various theorems are established concerning the correspondence between curves on S and on S', both for the general case and when S is contained in one or more linear complexes. If $(yz) \equiv g$, then $(\rho\sigma) \equiv g'$ is a generator of $H(g)$ of the second system, hence $H(g')$, $H(g)$ have g' in common. The residual intersection is therefore in general a space cubic, called the derivative cubic. It cuts g in two points, which with the two flecnodes on g make a harmonic range. Indeed, the idea of harmonic section, first found by Cremona to be made on the generators of certain surfaces by

the asymptotic lines, and somewhat extended by other writers, is here shown to permeate the whole theory. Conditions under which the derivative cubic is composite are determined, its linear complex found and a number of properties established. The osculating linear complex of g is introduced to prove and to generalize a number of known theorems on asymptotic lines.

In case H, H' have two lines in common ($\theta_4 = 0$) the residual intersection is the derivative conic. It cannot be composite unless S has a straight line directrix. Two conics belonging to consecutive generators cannot intersect. The developable formed by the plane of the derivative conic is discussed.

We next pass to the discussion of curves upon a surface. An arbitrary curve is one branch of the flecnode curve of an infinite number of ruled surfaces, but two curves chosen at random cannot in general form the complete flecnode curve of any surface. Similarly for the complex curve and for an asymptotic line.

Chapter XIII is concerned with a space curve, defined by a single equation of order 4. Its torsal cubic, osculating cubic, conic and linear complex are treated in detail, and a careful interpretation of the special tetrahedron of reference is given. A detailed comparison with the results of Halphen is then added, with application to anharmonic curves and certain plane curves.

Throughout the book copious references are given, and but few known theorems are left uncredited. A particularly commendable feature is the collection of exercises which follows each chapter; some of them are obvious corollaries of theorems just derived, others are less directly connected, and finally a liberal number indicated by a star are unsolved problems, containing suggestions for further investigation.

The book is provided with an index and is up to the usual standard of excellence maintained by Teubner. Although full of formulas, it is singularly free from typographical errors. Of the fifteen noticed by the reviewer, only two might cause confusion. On page 55, line 22, $M_{n/2}$ should be $M_{n-2/2}$, and on page 67, line 19, Ω_2 should be Ω_1.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
October, 1906.

SHORTER NOTICES.

Handbuch der Theorie der Cylinderfunktionen. Von NIELS NIELSEN, Privatdozent an der Universität Kopenhagen, Inspektor des mathematischen Unterrichts an den Gymnasien Dänemarks. Leipzig, Teubner, 1904. xii + 408 pp.

DR. NIELSEN'S treatise contains twenty-seven chapters, of which all but four are largely devoted to an exposition of his own researches. While many of the results are not new, he has given more than a score of new equations and over half as many generalizations of known theorems, also numerous new proofs. Some of these proofs are for theorems stated but not proved by Lommel, Hurwitz, Jacobi, H. F. Weber and others. In several cases his proofs are intended to supersede less rigorous proofs by Sonin, Mehler and Ermakoff. Thus the proof by Bourget, that $J^n(x) = 0$ and $J^{n+p}(x) = 0$, if n and p are integers, have no common root, is declared to be valid only for the case of multiple roots, a proof also criticised by Rayleigh.

A cylindrical harmonic is defined as a solution of two functional equations which are shown to lead to Bessel's equation. The general solution of the first fundamental equation is obtained, and from a new property of the second equation follows a solution in the form of a continued fraction. The influence which Kepler's equation has exerted upon the study of cylindrical harmonics is recognized, but the problem from which the functions obtained their name appears to have been overlooked. The author would advise dropping the name Bessel function, because Bessel used only integral parameters, also because Bernoulli, Euler, Laplace, Fourier, Poisson and others had previously known of them; but he has consented to call J^n the Bessel cylindrical harmonic and Y^n the Neumann cylindrical harmonic because of the fundamental work done by their respective investigators.

Dr. Nielsen has aimed to obtain generalized forms and theorems, and with this in view he has devoted considerable space in the first of the four parts of his work to Lommel's Π function and to the similar Φ function, thus laying a firm foundation for a new theory of definite integrals with cylindrical harmonics and for Schlömilch's series. In this part are

given also new developments of Krampe's integral $\int_0^x \epsilon^{-x^2} dx$, and two of $J^n(x)$.

A chapter is also given to the integration of Riccati's equation, an equation of the third order and one of the fourth order by methods which may be extended to those of higher order.

Although this treatise does not go into practical applications, the author deviates from this course by giving at the end of the first part a list and brief description of tables of Bessel's functions mostly J^0 and J^1, including one of J^n, also one of Y^0 and Y^1. He might have included a table by Dr. Meissel giving the first fifty roots of $J'(x) = 0$ and the corresponding maximum or minimum values of $J^0(x)$, which may be found in Gray and Mathews's Bessel Functions, together with the formulas by Professor J. McMahon in this connection.

The new theory of definite integrals mentioned above is given in the second part, and though it would have been possible to express the results as particular integrals of the general differential equation, the author has preferred not to do so, and has thus avoided introducing a large class of more general functions. Attention is called to an incorrect equation by Struve which is quoted by Gray and Mathews, page 238, example 46. The latter have made one correction, but the denominator of the integrand as printed should be raised to the nth power.

The third part is devoted to the development of analytic functions by means of cylindrical harmonics and gives a new analogy between Neumann's series of the first and second kind; also two new developments of elliptic integrals of the first kind. An interesting result is a new addition formula, $J^n(x + y)$ in an infinite series of terms each containing a product of two Bessel functions.

The development of arbitrary functions in terms of cylindrical harmonics is the topic of the fourth part. A series expansion by Lommel, proceeding according to Bessel functions of ascending orders, each differentiated the same number of times, does not appear. A new solution of Kepler's equation is given, believed to have advantages in practical application.

At the end of the book is a collection of more than sixty formulas or theorems under the headings: gamma functions, hypergeometric series, spherical harmonics, etc., also a few notes upon portions of the text. These theorems are referred to by number in order to avoid digressions in the course of the proofs. Here

appears a note concerning an important misprint in Dini's discussion of Fourier's series, referring also to a letter from Dini. Earlier in the text Dini was mentioned as the only one among several writers who had given a rigorous proof of the correctness of the expansion of $f(x)$ in a series of terms each involving a Bessel function. This expansion caused Todhunter to state that many German writers credit Fourier with its authorship, though in fact he did not give it; and into this category it may be inferred that Nielsen has fallen.

Last of all comes a most important part of the book, a very complete bibliography, giving references to both theoretical and applied work in cylindrical harmonics. In addition, at the bottom of many pages are references to the original sources of nearly all formulas, in many cases proved by methods different from those in the text. A paper by Glaisher on Riccati's equation appeared in *Philosophical Transactions* in 1881, not in 1882, while to Schläfli's credit may be added an extensive article in *Annali di Matematica*, series 2, volume 6. In 1867 Lommel mentioned nine writers, while in this list appear one hundred and fifty-five.

With all the work which Dr. Nielsen has brought within the compass of a volume of moderate size, and in which he has had so great a share, there remain unexplored fields. Apart from his frank statement that we do not know the necessary and sufficient conditions under which a function is developable in a Fourier's series, there are other topics more closely related to the text, such as the remainder terms in null developments, and the single valuedness of developments in a Schlömilch's series, also many topics not fully treated in the present work.

F. H. SAFFORD.

Space and Geometry in the Light of Physiological, Psychological and Physical Inquiry. By Dr. ERNST MACH. Translated from the German by THOMAS J. MCCORMACK. Chicago, The Open Court Publishing Company, 1906. 148 pp.

To appreciate this work it is necessary to view it in its relation to two complementary movements in modern mathematical thought, namely, the logical movement and another that may be significantly called biological. The aim of the former has been to detect and to enumerate all definite ideas or terms that are indefinable and all definite propositions that are indemonstrable,

to combine these primitives into all logically available sets of compatibles, and then by processes of pure inference to render explicit in systematic form the various contents implicit in the various sets. The method has been that of abstraction, postulation and deduction; the concern has been, not with external validity or applicability, but solely with consistence, with logical coherence, with conceptual and propositional harmony; and the effect has been more and more to eliminate intuition, to detach mathematics from experience, from reality, from life, from the sensuous world of things and events. The advance has followed two paths, the path of the well-known mathematical rigorists, and the path of symbolic logic under the leadership of C. S. Peirce, Schröder, and especially Peano; and these paths unexpectedly to most have been found to converge in the remarkable thesis that pure mathematics is symbolic logic and that pure logic is symbolized mathematics. A notable mark of the movement has been a strong tendency to nominalism, as witness, for example, Hilbert's Fundamental principles of geometry, in which the point, the line and the plane are nothing but names of undefined entities (replaceable by other names or entities or both) satisfying a prescribed system of postulates.

Meanwhile another movement has been going on and in recent years rapidly gaining in interest and force. It contains two principal components. These are easily confounded because they agree in seeking to reattach mathematics to experience, in demanding a reunion and a more intimate union than ever before of mathematical science and reality. Nevertheless the two components are entirely distinct. They differ radically in respect to the kind of union they severally contemplate. The one aims to establish or to reestablish a union supposed never to have existed or to have been broken up. The other aims to discover a union supposed to have existed always as in the nature of things. The former is partly due to a reaction against the logical movement, the tendency to nominalism, and partly to the increasing demand of an increasing number of sciences such as engineering, chemistry, anthropology, economics, psychology and statistics, that mathematics shall cease to dwell apart, interested exclusively in its own evolution as a pure science, and adapt itself, its teaching and investigation, to their needs. The latter component is not of the nature of such reaction and is not due to such a demand. Its aim is to unite mathematics and experi-

ence, not in the sense of rendering the science applicable in other fields of investigation, but in the sense of showing that mathematical concepts, however tenuous or pure or remote or recondite, have been literally evolved continuously in accordance with the needs of the animal organism out of the elements (feelings) of physiological experience. This is why the second movement may properly be described as biological. The extent to which the movement may be destined to succeed is a question that only time can answer. The enterprise is undoubtedly legitimate and is one that men as rational beings were bound sooner or later to undertake. The significance of it, its bearings on theory and on practice, on teaching in particular, and on the conception that the future mathematician may have of his science and of its relations to other modes and forms of intellectual activity, can scarcely fail to be profound.

The leading contributor to the biological movement in mathematical criticism, at least the most widely known contributor, is Professor Mach, whose Science of mechanics, Popular scientific lectures, and Contributions to the analysis of the sensations are well-known and are highly valued in scientific and critical circles everywhere throughout the western world. The book in hand ought to be read and pondered by every teacher of mathematics and by every educated guardian of the young. Physiological space, the space of vision, the space of touch, the space of audition, metrical space, the space of geometry, the correspondences, likenesses and differences of these, feelings of constance, of motion, of velocity, of acceleration, of locations, the interplay and biological functions of these sensations, their contributions to geometry,— such are some of the themes of the discussion, which, running from simple facts of vision through a critical characterization and comparison of the chief varieties of metageometry, is remarkable alike for the questions it propounds and for the answers it gives or suggests. The Kantian philosopher will find here ample reason to reconsider his master's doctrine of space, the famous saying of Kant that "Thoughts without contents are empty, intuitions without concepts are blind" being happily transformed by Mach into "Concepts without intuitions are blind, intuitions without concepts are lame." The indications are clear that the psychologist of the future will, contrary to the rash predictions of Dr. Hall, find a rich field for psychological research in the concepts of mathematics; and the mathematician in his turn will not fail

to find evidence that the purest offspring of his thought may trace a legitimate lineage back and down to the rudiments of physical and physiological experience. The author's discourse carries waters from numerous confluent sciences, and is a living witness to the unity of knowledge.

The translation, as one knowing Mr. McCormack's previous work would expect, is well-nigh perfect. The Open Court Company is again to be congratulated on its excellent judgment and on its generosity in the service of science.

C. J. KEYSER.

NOTES.

WITH the present issue of the BULLETIN Professor H. S. WHITE retires from the editorial staff, to assume larger responsibilities as member of the Editorial Committee of the *Transactions*. The Committee of Publication takes this opportunity to express its sense of obligation to Professor White for his valuable editorial support during the past two years.

A MEETING of mathematicians interested in the formation of a Southwestern Section of the AMERICAN MATHEMATICAL SOCIETY was held at Columbia, Mo., on Saturday, December 1, about thirty-five persons being in attendance. Over twenty papers were presented. Steps were taken toward permanent organization. A full report of the meeting will appear in the BULLETIN.

THE concluding (October) number of volume 28 of the *American Journal of Mathematics* contains the following papers: "Functions of three independent variables (concluded)," by H. L. COAR; "An invariant condition for certain automorphic algebraic forms," by A. B. COBLE; "On some cases of motion of a solid in infinite liquid," by G. KOLOSOFF; "On the arrangement of the real branches of plane algebraic curves," by V. RAGSDALE.

THE January number (volume 8, number 2) of the *Annals of Mathematics* contains: "Circles orthogonal to a given sphere," by C. L. E. MOORE; "On functional determinants," by PAUL SAUREL; "Involutory transformations in the projective group and its subgroups," by E. B. WILSON; "On the convergence and differentiation of certain classes of trigonometric series," by W. C. BRENKE; "Note on the definition of an abelian group by independent postulates," by W. A. HURWITZ.

AT the annual meeting of the London mathematical society held on November 8, the following officers were elected: W. BURNSIDE, president; A. E. H. LOVE and J. H. GRACE, secretaries; J. LARMOR, treasurer; two vice-presidents and ten other members of the council. The following papers were read: By A. R. FORSYTH (presidential address), "Partial differential equations, some criticisms and some suggestions";

by A. C. DIXON, "Harmonic expansions of functions of two variables"; by G. N. WATSON, "General solution of Laplace's equation in n dimensions"; by H. HILTON, "On subgroups of a finite abelian group"; by J. E. CAMPBELL, "On Bäcklund's transformation and the partial differential equation $s = F(x, y, z)$"; by H. BATEMAN, "Inversion of a double integral."

THE annual meeting of the National academy of sciences was held at Harvard University, November 20–21.

THE seventh meeting of the Association of teachers of mathematics in the Middle States and Maryland was held at the Central high school, Philadelphia, on December 1, 1906. After an address of welcome by R. E. THOMPSON, president of the Central high school, the following papers were read: By CHARLOTTE F. MCLEAN: "Some suggestions for relieving the strain in elementary mathematics"; by HARRY ENGLISH: "The teacher—his preparation, place, and power'; by A. G. RAU: "Co-ordination in mathematics"; by JOHN MILLER: "The fundamental theorems and elementary mathematics"; by ALICE M. MCKELDEN: "The problems that arise in the teaching of elementary algebra"; by F. H. SAFFORD: "Inversion."

The following officers were elected for the coming year: president, E. S. CRAWLEY; vice-president, J. S. FRENCH; secretary-treasurer, J. T. RORER.

THE Central association of science and mathematic teachers, acting with the American society of teachers of mathematics and the natural sciences, has issued a call for a meeting of delegates of associations of like nature, to be held at Columbia University on December 27, for the purpose of discussing the question of the formation of a national federation of teachers of mathematics and science.

THE annual meeting of the New York state science teachers association was held at Teachers College, Columbia University, New York, on December 26–27. Several papers on mathematical topics were presented.

THE following books are announced in the press of B. G. Teubner, Leipzig, and will probably appear in a few weeks: Encyklopädie der Elementar-Mathematik, volume 3, by H.

WEBER and J. WELLSTEIN; Vorlesungen über Geschichte der Mathematik, volume 4 (1758–1800), by M. CANTOR; Theorie des Integrallogarithmus und verwandter Transzendenten, by N. NIELSEN; Vorlesungen über die Elemente der Differential- und Integralrechnung, by H. BURKHARDT; Vorlesungen über Zahlentheorie, by J. SOMMER; Fragen der Elementargeometrie, volume 2, by F. ENRIQUES, translated by H. FLEISCHER; Synthetische Geometrie der Kegelschnitte, by P. SCHAFHEITLIN.

AMONG the new models which have just been added to the list of Schilling in Halle are an elliptic circle, by K. ROHN; a wooden model to illustrate the Dandelin theorem, by E. KÖTTER, and a metallic one to demonstrate the generation of an ellipse in space, by C. HILDEBRANT; three dimensional nets of four dimensional bodies, by R. GAETSCHENBERGER; and a plaster model of the locus of the center of a chord of a twisted quartic curve, by D. BÖHMLÄNDER.

THE new rules regarding the mathematical tripos of Cambridge University, as sanctioned by the senate on October 25 are as follows: (1) A student may be a candidate for part I at a date not earlier than the second term nor later than the seventh term; (2) A student having failed to obtain honors may try a second time, upon accepted recommendation; (3) The subjects must be chosen from the schedule annexed to the report; (4) The list of successful candidates will be published in three classes, each arranged alphabetically; (5) The subjects for part II must be from A, B of report, and part I; (6) The successful candidates will be announeed as wranglers, senior optimes and junior optimes, each arranged alphabetically; (7) The class will be determined by the proficiency shown in A, provided a fair showing is made in B. (Cf. BULLETIN, volume 12, page 468 and volume 13, page 131.)

OXFORD UNIVERSITY (*Hilary term*, 1907).—By Professor W. ESSON: Comparison of analytic and synthetic methods in the theory of conics, two hours; Synthetic properties of cubics, one hour.—By Professor E. B. ELLIOTT: Elements of elliptic functions, two hours; Theory of numbers (continued), one hour.—By Professor H. H. TURNER: Mathematical astronomy, two hours.—By Professor A. E. H. LOVE: Theory of potential, two hours; Calculus, two hours.—By Mr. P. J.

KIRKBY: Higher plane curves, two hours. — By Mr. A. L. DIXON: Calculus of finite differences, two hours. — By Mr. J. E. CAMPBELL; Differential equations, II, two hours. — By Mr. C. H. SAMPSON: Solid geometry, two hours. — By Mr. J. W. RUSSELL: Determinants, two hours. — By Mr. C. LEUDESDORF: Geometry of inversion, two hours. — By Mr. A. E. JOLLIFFE: Analytical geometry, two hours. — By Mr. R. F. MCNEILE: Integral calculus, two hours. — By Mr. E. H. HAYES: Elementary mechanics, three hours. — By Mr. C. H. THOMPSON: Dynamics of a particle, two hours.

AMONG the medals presented at the annual meeting of the Royal society of London was a royal medal to Professor A. G. GREENHILL for his researches in the applications of elliptic functions.

AT the Stuttgart meeting of the Deutsche Mathematiker-Vereingung the sum of 933.80 marks was set apart to restore and preserve the grave of RIEMANN, who was buried at Biganzolo, Italy, in 1866.

AT the University of Maine, Mr. E. E. MOOTS has been appointed instructor in mathematics.

CATALOGUES of second-hand mathematical works: Gustav Fock, Schlossgasse 7, Leipzig, Germany, Antiquariats-Verzeichnis no. 293, 4401 titles in mathematics and physics. — W. Junk, Kurfüstendamm 201, Berlin, Bulletin nos. 1–2, 163 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAKER (W. M.). Algebraic geometry. New treatise on analytical conic sections. London, 1906. 8vo. 338 pp. Cloth.

BOURDON. Application de l'algèbre à la géométrie, comprenant la géométrie analytique à deux et à trois dimensions. 9e edition. Paris, Gauthier-Villars, 1906. 8vo. 18 + 650 pp. F. 9.00

BROMWICH (T. J. I'A.). Quadratic forms and their classification by means of invariant factors. Cambridge, University Press, 1906. 8vo. 8 + 100 pp. (Cambridge Tracts in mathematics and mathematical physics, No. 3.) 3s. 6d.

COHEN (A.). Differential equations. Boston, Heath, 1906. 8vo. Half leather. $2.00

CZUBER (E.). Vorlesungen über Differential- und Integralrechnung. (In 2 Bänden.) Vol. II. 2te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1906. 8vo. 8 + 528 pp. Cloth. M. 12.00

DEMUTH (W.). Ueber unendliche Potenzreihen, deren Koeffizienten nach arithmetischen Reihen höherer Ordnung fortschreiten. (Progr.) Blankenburg, 1906. 4to. 20 pp.

DURÈGE (H.). Elemente der Theorie der Funktionen einer komplexen veränderlichen Grösse. 5te Auflage, neubearbeitet von L. Maurer. Leipzig, Teubner, 1906. 8vo. 10 + 397 pp. M. 9.00

EHRAT (J.). Ueber das Algebraischwerden der Integrale irrationaler Differentiale von der Form $\pi(x, y)dx$, in welchen $\pi(x, y)$ eine rationale Funktion ist und zwischen x und y eine allgemeine Gleichung zweiter Ordnung besteht. (Diss.) Zürich, 1906. 8vo. 50 pp. M. 2.50

ENRIQUES (F.). Problemi della scienza. Bologna, 1906. 8vo. 593 pp. L. 10 00

GOTTSCHALK (A.). Zur Integration der Eulerschen Differentialgleichungen. (Progr.) Gronau, 1906. 8vo. 7 pp.

GRÜNWALD (A.). Betrachtung von Fusspunktskurven in der Ebene und im Raume. (Progr.) Prag, 1906. 8vo. 34 pp.

HASS (E.). Merkwürdige Kurven im Dreieck und ihre Beziehung zu den sogenannten "merkwürdigen Punkten." (Progr.) Duisburg, 1906. 4to. 27 pp.

ISENKRAHE (C.). Ueber die zweiunddreissig Lösungsergebnisse des erweiterten Malfattischen Problems. (Progr.) Trier, 1906. 8vo. 46 pp.

JUEL (C). Om ikke-analytiske kurver. Kjöbenhavn, 1906. 4to. 78 pp. Kr. 1.95

JUNKER (F.). Höhere Analysis. Teil I: Differentialrechnung. Leipzig, Göschen, 1906. 16mo. 204 pp. (Sammlung Göschen.) M. 0.80

KIENAST (A.). Ueber die Darstellung der analytischen Funktionen durch Reihen, die nach Potenzen eines Polynoms fortschreiten und Polynome eines niederen Grades zu Koeffizienten haben. (Diss.) Zürich, 1906. 8vo.

KISTLER (H.). Ueber Funktionen von mehreren komplexen Veränderlichen. (Diss., Gottingen.) Basel, 1905. 8vo. 45 pp.

LESCHANOWSKY (H.). Gemeinverständliche erste Einführung in die höhere Mathematik und deren Anwendung. Wien, Fromme, 1906. 8vo. 8 + 85 pp. M. 2.50

LÜTKEMEYER (G.). Der analytische Charakter der Integrale von partiellen Differentialgleichungen zweiter Ordnung in Anwendung auf die Theorie der Flächenverbiegung. (Progr.) Gelsenkirchen, 1906. 4to. 29 pp.

MAILLET (E.). Introduction à la théorie des nombres transcendants et des propriétés arithmétiques des fonctions. Paris, Gauthier-Villars, 1906. 8vo. 5 + 280 pp. Fr. 10.00

MAUDERLI (S.). Die Interpolation und ihre Verwendung bei der Benutzung und Herstellung mathematischer Tabellen. (Progr.) Solothurn, 1906. 8vo. 146 pp.

MAURER (L.). See DURÈGE (H.).

MILLER (G. A.). Groups generated by two operators which transform each other into the same power. 8vo. 4 pp. (*Prace matematyczno-fizyczne*, vol. 17, pp. 119–122.)

MÜLLER (F. J.). Abbildung eines Sphäroidstreifens auf die Ebene. (Diss.) Würzburg, 1906. 8vo. 27 pp.

MÜNCHEN (K.). Die Krümmungslinien auf der windschiefen Schraubenfläche. (Progr.) Feldkirch, 1906. 8vo. 47 pp.

NIEWENGLOWSKI (G. N.). Les mathématiques de la médecine. Paris, 1906. 8vo. F. 2.50

OCTAVIO DE TOLEDO (L.). Programa y cuestionario de análisis matemático. Primer curso. Madrid, Fortanet, 1906. 40 pp. P. 1.00

SCHEFFERS (G.). See SERRET (J. A.).

SCHUBERT (H.). Auslese aus meiner Unterrichts- und Vorlesungspraxis. Vol. III. Leipzig, Göschen, 1906. 8vo. 250 pp. Cloth. M. 4.00

SERRET (J. A.). Lehrbuch der Differential- und Integralrechnung. Nach Axel Harnacks Uebersetzung. (In 3 Bänden.). 3te Auflage, neu bearbeitet von G. Scheffers. Vol. I: Differentialrechnung. Leipzig, Teubner, 1906. 8vo. 16 + 624 pp. M. 12.00

STECKELBERG (H.). Die Elemente der Differential- und Integralrechnung. (Progr.) Witten, 1906. 8vo. 48 pp.

STURM (A.). Geschichte der Mathematik. Leipzig, Göschen, 1906. 16mo. 152 pp. (Sammlung Göschen.) M. 0.80

WEINSTEIN (B.). Die philosophischen Grundlagen der Wissenschaften. Vorlesungen, gehalten an der Universität Berlin. Leipzig, Teubner, 1906. 8vo. 14 + 544 pp. Cloth. M. 9.00

WEITBRECHT (W.). Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Leipzig, Göschen, 1906. (Sammlung Göschen.) M. 0.80

II. ELEMENTARY MATHEMATICS.

ANGLADA Y PÉREZ (D. C.). Tratado de aritmética y algebra. Madrid, Marzo, 1906. 8vo. 271 pp.

BALTIN (R.) und MAIWALD (W.). Kurzgefasstes Lehrbuch der Mathematik für Seminare und Präparandenanstalten. Unter Zugrundelegung des Lehrbuches von H. Müller "Die Mathematik auf den Gymnasien und Realschulen," Teil IB, nach den Lehrplänen von 1901 für Seminare usw. bearbeitet. 2te, verbesserte Auflage. Leipzig, Teubner, 1906. 8vo. 10 + 218 pp. Boards. M. 2.40

BARDEY (E.). Aufgabensammlung, methodisch geordnet, mehr als 8000 Aufgaben enthaltend über alle Teile der Elementar-Arithmetik, vorzugsweise für Gymnasien, Realgymnasien und Oberrealschulen. In alter und neuer Ausgabe. Alte Ausgabe. 28ste Auflage. Leipzig, Teubner, 1906. 8vo. 14 + 330 pp. Boards. .M. 3.20

——. Arithmetische Aufgaben nebst Lehrbuch der Arithmetik, vorzugsweise für höhere Bürgerschulen, Realschulen, Progymnasien und Realprogymnasien. 14te Auflage, in der neuen Rechtschreibung. Leipzig, Teubner, 1906. 8vo. 10 + 269 pp. Boards. M. 2.40

BOUVART (C.) et RATINET (A.). Nouvelles tables de logarithmes à cinq décimales. Table numérique; table trigonométrique. Division centésimale établie conformément à l'arrêté ministériel du 3 août 1901, à l'usage des candidats au baccalauréat et aux écoles polytechniques et de Saint-Cyr. 5e édition. Paris, Hachette, 1906. 8vo. 128 pp. Fr. 2.00

BRUHNS (C.). Neues logarithmisch-trigonometrisches Handbuch auf sieben Decimalen. Leipzig, Tauchnitz, 1906. 8vo. 24 + 610 pp. M. 4.20

CRANTZ (P.). Arithmetik und Algebra. Leipzig, Teubner, 1906. 8vo.

DOMBROVSKI (A.). Pri novaj trigonometriaj sistemoj. Originale verkis. Berlin, Möller, 1906. 8vo. 35 pp. M. 1.50

DUPUIS (J.). Tables de logarithmes à sept décimales. Edition stéréotype contenant les logarithmes des nombres de 1 à 100,000, les logarithmes des sinus et des tangentes des angles calculés de seconde en seconde pour les cinq premiers degrés et de dix secondes en dix secondes pour tous les degrés du quart de cercle et quelques tables usuelles. 13e tirage. Paris, Hachette, 1906. 8vo. F. 8.00

F. I. C. Tables de logarithmes à cinq décimales pour les nombres de 1 à 10,000 et pour les functions trigonométriques, de minute en minute. Paris, Poussielgue, 1907. 16mo. 8 + 148 pp.

F. G. M. Cours d'algèbre élémentaire, conforme aux derniers programmes de l'enseignement secondaire (1902). Paris, Poussielgue, 1906. 16mo. 8 + 568 pp.

FERVAL (H.). Eléments de trigonométrie rédigés conformément aux programmes de l'enseignement secondaire et de l'enseignement primaire supérieur, contenant 556 exercices et problèmes. 2e édition. Paris, Hachette, 1906. 16mo. 299 pp. F. 2.50

FONT ET HUGON. Supplément au cours d'algèbre (programme de 1905). Mâcon, Protat, 1906. 16mo. 153 pp.

GRÉVY (A.). Géométrie plane, à l'usage des élèves des classes de seconde C et D (programme du 27 juillet 1905). 2e édition. Paris, Vuibert, 1907. 8vo. 296 pp. F. 3.00

——. Géométrie théorique et pratique. 3e édition. Paris, Vuibert, 1907. 16mo. 8 + 281 pp.

GUBLER (E.). Aufgaben aus der allgemeinen Arithmetik und Algebra für Mittelschulen. Drittes Heft: Resultate und Auflösungen. Zürich, 1906. 8vo. 35 pp. Boards. M. 1.50

GUTZMER (A.). See REFORMVORSCHLÄGE.

HAMILTON (J. G.) and KETTLE (F.). Second geometry book. London, Arnold, 1906. 12mo. 300 pp. 3s. 6d.

HOLL (W.). Lehrbuch der Geometrie. Die Lehre von den geometrischen Raumgrössen in geeigneter Verbindung mit Zeichnen und Rechnen für niedere landwirtschaftliche Lehranstalten, gewerbliche Fortbildungs- und Mittelschulen. 5te Auflage. Stuttgart, Kohlhammer, 1906. 8vo. 8 + 228 pp. Boards. M. 2.00

JULING (G.). Fünfstellige Logarithmen-Tafeln für Schüler. 2te Auflage. Leipzig, Berger, 1906. 8vo. 152 pp. Cloth. M. 1.20

KNOTT (C. G.). Four-figure mathematical tables. New York, Van Nostrand, 1906. 12mo. 24 pp. Cloth. $0.25

LEHNEN (W.). Teilung eines jeden gegebenen Winkels in den Primzahlen 3, 5, 7, 11, 13, usw. entsprechende Teile. Leipzig, Teubner, 1906. 8vo. 2pp.

LEIGHTON (A.). Elementary mathematics. Algebra and geometry. London, Blackie, 1906. 12mo. 304 pp. Cloth. 2s.

LONDON University intermediate mathematics papers. London, Clive, 1906. 12mo. 96 pp. 2s.

MAIWALD (W.). See BALTIN (R.).

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